Polynomial homological Dehn functions from non-proper actions

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Theorem Let X be an $(n-1)$-acyclic cocompact locally finite G-simplicial complex.

Assume that

- $FV_X^i \preceq$ polynomial for $2 \le i \le n$

for each k-cell σ

- G_σ is of type FP_{n-d}
- $FV_{G_\sigma}^i \preceq$ poly. for $2 \le i \le n-k$
- G_σ is at most polynomially distorted in G $\left[\operatorname{dist}_{G_\sigma}^G(n) := \sup \{ \ell_{G_\sigma}(h) \mid h \in G_\sigma, \ell_G(h) \le n \} \preceq \text{poly} \right]$

Then G is of type FP_n and $FV_G^i \preceq$ poly. for $2 \le i \le n$.

Definition: Let (C_*, ∂_*) be a chain complex of free $\mathbb{Z}$-modules.

On C_i consider the ℓ^1-norm $\left\| \sum_b a_b b \right\| := \sum_b |a_b|$.

For a cycle $c \in \ker(\partial_i)$ define

- $FV(c) := \min \{ \|b\| \mid b \in C_{i+1}, \partial b = c \}$
- $FV_{C_*}^i(v) := \sup \{ FV(c) \mid c \in \ker(\partial_{i-1}), \|c\| \le v \}$

For a simplicial complex X: $FV_X^i := FV_{C_*(X)}^i$

Let G be of type $FP_n \iff \exists$ free projective $\mathbb{Z}G$-resolution

$$\dots \to P_n \to P_{n-1} \to \dots \to P_0 \twoheadrightarrow \mathbb{Z}$$

s.t. P_i is f.g. for $i \le n$

Define $FV_G^n := FV_{P_*}^n$

Remark: Up to asymptotic equivalence FV_G^n is well-defined and a QI-invariant.

$$f \preceq g :\iff \exists C \text{ s.t. } f(x) \le C g(Cx + C) + Cx + C$$
$$f \approx g :\iff f \preceq g \text{ and } g \preceq f.$$

- All definitions (and results) work in more generality for rings with certain norms.

Why care about $FV_G^i \leq \text{poly}$?

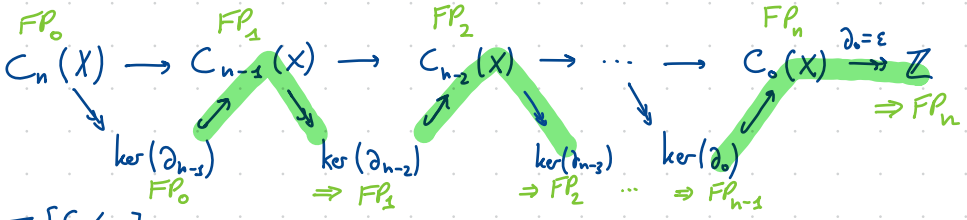
If $FV_G^i \leq \text{poly}$, then $H^i(G;V) \xleftarrow{\cong} H^i_{pol}(G;V) \leftarrow H^i_b(G;V)$

is an isomorphism for $i \leq n$. homology of $C^k_{pol}(G,V) = \{f : G^{k+1} \xrightarrow[G\text{-equiv.}]{} V \mid \|f(1,-,g_k)\|_V \leq c \cdot (1 + \sum l_G(g_i))^d \text{ for some } c,d\}$

How to prove such a theorem?

① How to prove FP_n?

Consider



$C_k(X) \cong \bigoplus_{\sigma \in I_k} \mathbb{Z}[G/G_\sigma]$

Know G_σ is of type FP_{n-k}

Induction

$\exists \quad \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} P^\sigma_{n-k} \to \cdots \to \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} P^\sigma_0 \to \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} \mathbb{Z} \cong \mathbb{Z}[G/G_\sigma]$

proj. $\mathbb{Z}G_\sigma$-resolution, P^σ_i f.g. for $i \leq n-k$

$\Rightarrow C_k(X)$ is of type FP_{n-k}

Extension

If $0 \to A \to B \to C \to 0$ is a s.e.s of $\mathbb{Z}G$-modules

then $A\ FP_{n-1} + B\ FP_n \Rightarrow C\ FP_n$

② For polynomial Dehn functions:

$\exists$ properties $FP_n(E_G)$ of $\mathbb{Z}G$-modules satisfying

- Induction: If $H \leq G$ is at most polynomially distorted:

  M type $FP_n(E_H) \Rightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ type $FP_n(E_G)$

- Extension: $0 \to A \to B \to C \to 0$ for certain s.e.s. (*)

  $FP_{n-1}(E_G) + FP_n(E_G) \Rightarrow FP_n(E_G)$

  (satisfied by $\ker(\partial) \to C_i(X) \to \ker(\partial)$ under assumption of Theorem)

- $\mathbb{Z}$ is $FP_n(E_G) \iff G$ is of type FP_n and $FV_G^i \leq \text{poly}$ for $2 \leq i \leq n$.

Definition of $FP_n(E_G)$: Let M be of type FP_n and

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M$$

(h_1, h_0, h_{-1} the contracting homotopies)

be a free $\mathbb{Z}G$-resolution with P_i f.g. for $i \leq n$.

Def: M is $FP_n(E_G)$ if there is a $\mathbb{Z}$-linear chain contraction $h: P_i \to P_{i+1}$ s.t. $\mathbb{Q} \otimes h_i$ is bounded for $i \leq n-1$.

w.r.t the Cayley Bornology on $\mathbb{Q}G$-modules

On $\mathbb{Q}G$ consider the family of norms

$$\| \textstyle\sum a_g g \|_k^G := \sum |a_g| (1 + \ell_G(g))^k$$

$\rightsquigarrow$ Induces corresponding ℓ^1-norms on free $\mathbb{Q}G$-modules.

Say $B \subseteq \mathbb{Q}G^n$ is B_G-bounded if it is bounded w.r.t. $\|-\|_k^G$ $\forall k$.

A $\mathbb{Q}$-linear map $f: \mathbb{Q}G^n \to \mathbb{Q}G^m$ is bounded if it maps B_G-bounded sets to B_G-bounded sets

Remark: · This can be extended to all $\mathbb{Q}G$-modules

· If f is equivariant, then f is automatically bounded.

(*) $0 \to A \to B \to C \to 0$ admitting a $\mathbb{Z}$-linear splitting r, s s.t. $\mathbb{Q} \otimes r$, $\mathbb{Q} \otimes s$ are bounded

- Why use $\|\cdot\|_k^G$? There is no ℓ^1-norm bounded contraction $h_0 : \mathbb{Q} \otimes P_0 \to \mathbb{Q} \otimes P_1$ if G is infinite

  w.l.o.g $P_0 = \mathbb{Z}G$. Then $\partial h_0(g-1) = g-1$ for every $g \in G$

  but $\|g-1\| = 2$

  $h_0(g-1) \hat{=}$ path from g to 1, so $\|h_0(g-1)\| \approx \ell_G(g)$.

- Relation between $C^k_{pol}(G;V)$ and $\|\cdot\|_k^G$.

  $f : G^{k+1} \to V$ equivariant $\rightsquigarrow$ $\overline{f} : (\mathbb{R}G)^k \to V$ $\mathbb{R}G$-linear. $\overline{f}($

  $f(1, g_1, \dots, g_k) \leq C \cdot (1 + \sum \ell_G(g_i))^d$ $\quad$ $\|f(x)\|_V \leq C(\|f(x)\|_d^G)^k$ $\forall x \in (\mathbb{R}G)^k$

  for some C, d $\quad$ for some C, d

- How to see $\mathbb{Z}$ $FP_n(E_G) \Longrightarrow FV_G^i \leq$ poly $2 \leq i \leq n$.

  Define $wFV^i_{C_*}$ just as $FV^i_{C_*}$ with $\|\cdot\|_1^G$ instead of $\|\cdot\| = \|\cdot\|_0^G$.

  Let $h_i : P_i \to P_{i+1}$ be a bounded contraction, $c \in P_i$ a cycle.

  Then $\partial h_i(c) = c$. h_i bounded $\Rightarrow \exists\, C, d : \|h_i(x)\|_1^G \leq \|x\|_d^G \leq (\|x\|_1^G)^d$, $x \in P_i$

  $\Longrightarrow$ $wFV^i_{P_*} \approx wFV^i_G \leq$ poly.

  We also have $wFV^i_G \leq$ poly $\Longleftrightarrow$ $FV^i_G \leq$ poly for $i \geq 2$.