QUASI-ISOMETRY INVARIANCE OF
DISCRETE HIGHER FILLING FUNCTIONS

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ABSTRACT. We prove that homological filling functions over a ring R equipped with the discrete norm are quasi-isometry invariants for all groups of type FP_n. This confirms a conjecture of Bader–Kropholler–Vankov in the case of discrete norms. The proof uses a technique of equipping free chain complexes with a geometric structure, allowing for analogues of cellular constructions in the purely algebraic setting. As a further application we prove quasi-isometry invariance for a weighted version of integral and discrete filling functions originally introduced in the study of the rapid decay property.

1. INTRODUCTION

For a finitely presented group Γ, the classical *Dehn function* $\delta_\Gamma(l)$ quantifies the difficulty of solving the word problem by measuring the maximal filling area of a null-homotopic word of length at most l. From a computational viewpoint, δ_Γ reflects the complexity of a brute-force approach to the word problem in Γ. This extends naturally to higher dimensions: replacing loops by n-spheres yields the *homotopical* higher Dehn functions δ_Γ^n. It is well known that these functions are quasi-isometry invariants for groups of type F_n, that is, groups admitting a classifying space with finite n-skeleton [AWP99].

Instead of filling spheres, one can also study the maximal filling volume of integral n-cycles, giving the *homological* higher filling functions $\mathrm{FV}_{\mathbb{Z},\Gamma}^{n+1}$. Homological filling functions have the advantage that they readily generalise to arbitrary normed coefficient rings. This yields a large family of functions $\mathrm{FV}_{R,\Gamma}^n$ depending on both the choice of coefficients and on the chosen norm [LM22, BKV25].

The most common coefficient choices for studying finiteness properties and filling functions – besides the integers with the Euclidean norm – are finite fields equipped with the discrete norm, where the size of a chain is the cardinality of its support [Kro21, LM22, KK21]. More generally, any ring can be given the discrete norm, and we denote the resulting *discrete* filling function by $\mathrm{DFV}_{R,\Gamma}^n$.

A natural question is whether the homological filling functions $\mathrm{FV}_{R,\Gamma}^n$ are also quasi-isometry invariants of the group. In the case of integral filling functions $\mathrm{FV}_{\mathbb{Z},\Gamma}^n$ this is known to be true for groups of type F_n due to Fletcher [Fle98] and Young [You11]. Recent work of Bader, Kropholler, and Vankov [BKV25] shows that $\mathrm{FV}_{R,\Gamma}^n$ is a quasi-isometry invariant for groups of type $\mathrm{FH}_n(R)$, provided the functions take only finite values for both groups. Moreover, they proved that if a

group Γ is of type $\mathrm{FH}_n(R)$, then $\mathrm{DFV}_{R,\Gamma}^n$ is finite. They conjectured that a similar statement should hold for all groups of type $\mathrm{FP}_n(R)$ and arbitrary coefficients.

Conjecture (Bader–Kropholler–Vankov)**.** Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{FV}_{R,G}^n \approx \mathrm{FV}_{R,H}^n$.

Note that it is a well-known open problem, whether $\mathrm{FP}_n(R)$ and $\mathrm{FH}_n(R)$ are equivalent for $n > 2$.

Discrete filling functions. In this work, we prove the conjecture of Bader–Kropholler–Vankov for discrete filling functions, completing the picture of quasi-isometry invariance for all standard coefficient choices.

Theorem 5.5. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then* $\mathrm{DFV}_{R,G}^n \approx \mathrm{DFV}_{R,H}^n$.

We deduce Theorem 5.5 by first proving finiteness of discrete filling functions for groups of type $\mathrm{FP}_n(R)$.

Theorem 4.4. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Then $\mathrm{DFV}_{R,\Gamma}^n$ takes finite values.*

Theorem 5.5 then follows from a generalisation of the criterion given by [BKV25].

Theorem 5.4. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Assume* $\mathrm{FV}_{R,G}^i(l), \mathrm{FV}_{R,H}^i(l) < \infty$ *for all $2 \leq i \leq n$ and all l. Then* $\mathrm{FV}_{R,G}^n \approx \mathrm{FV}_{R,H}^n$.

As it is already known that $\mathrm{FV}_{\mathbb{Z},\Gamma}^n$ is finite for all groups of type $\mathrm{FP}_n(\mathbb{Z})$ with $n \geq 2$, this theorem also implies quasi isometry invariance for integral filling functions in the non-finitely presented case.

Weighted filling functions. The techniques used in the proof of Theorem 5.4 generalise to a *weighted* variation of homological filling functions. Any free $R\Gamma$-module $L = \bigoplus_{b \in B} R\Gamma b$ carries an ℓ^1-norm obtained by viewing it as a free R-module with basis ΓB and setting $\left\| \sum a_{\gamma b} \gamma b \right\| = \sum |a_{\gamma b}|$. In the weighted variant, each basis element γb is assigned the weight $1 + \ell_\Gamma(\gamma)$, where ℓ_Γ denotes the word length in Γ. The resulting norm $\|\cdot\|^\Gamma$ and the corresponding weighted filling functions $\mathrm{wFV}_{R,\Gamma}^n$ appear naturally in several situations. They were first studied by Ogle [Ogl05] and Ji–Ramsey [JR09] in connection with the Rapid Decay property. They were also utilised by Bader–Sauer in their work on higher property (T) [BS23] to deduce isomorphisms in polynomial cohomology under the assumption that $\mathrm{wFV}_{\mathbb{Z},\Gamma}^n$ is polynomially bounded. Furthermore, weighted filling functions are used in upcoming joint work with Roman Sauer to give a characterization for polynomiality of filling functions using a homological algebra framework [SW26].

Our techniques for non-weighted filling functions extend to this setting, yielding weighted versions of Theorems 5.4 and 5.5.

Theorem 5.9. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then* $\mathrm{wDFV}_{R,G}^n \approx \mathrm{wDFV}_{R,H}^n$.

Theorem 5.10. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(\mathbb{Z})$. Then* $\mathrm{wFV}_{\mathbb{Z},G}^n \approx \mathrm{wFV}_{\mathbb{Z},H}^n$.

Proof strategy. The key step in the proof of Theorem 4.4 is a translation of geometric arguments on cellular chain complexes into an algebraic framework for free $R\Gamma$-chain complexes. This approach, developed in Section 3, replaces cellular subcomplexes with finitely generated R-subcomplexes and reproduces the relevant geometric constructions purely algebraically. Conceptually, under mild hypotheses (which $\mathrm{FP}_n(R)$-resolutions can be arranged to satisfy), free $R\Gamma$-chain complexes behave much like the cellular chain complexes of genuine spaces.

The resulting techniques appear interesting in their own right and may admit further applications. Indeed, an easy application of this framework is Proposition 4.1, which provides another proof of the finiteness theorem of [FMP18] for the integral filling functions. We also present a new proof of the quasi-isometry invariance of cohomology with coefficients in the group ring for groups of type $\mathrm{FP}(R)$.

Once we developed this algebraic framework, the proof of Theorem 4.4 is a generalisation of the corresponding geometric proof in [BKV25] to this setting.

Structure. The paper is organised as follows. In Section 2 we recall the definition of filling functions over normed rings and establish basic properties needed throughout. Section 3 develops the algebraic counterpart of geometric arguments on Γ-complexes, which is then applied in Section 4 to prove the finiteness of discrete filling functions. Section 5 gives the proof of Theorem 5.4 and its weighted counterpart. Finally, Section 6 contains the proof of the quasi-isometry invariance of cohomology using the techniques from Sections 3 and 5.

2. Homological filling functions over normed rings

Following [BKV25] and [KK21], we give a short introduction to homological filling functions over arbitrary normed rings.

Throughout, let Γ be a group and R a ring with unit $1 \neq 0$. All modules considered are left modules unless indicated otherwise.

2.1. **Normed rings.**

Definition 2.1. A *norm* on a ring R is a function $|\cdot| : R \to \mathbb{R}_{\geq 0}$ such that

(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
(ii) $|x + y| \leq |x| + |y|$;
(iii) $|xy| \leq |x|\,|y|$.

A ring together with a choice of norm is called a *normed ring*. Two norms $|\cdot|_1, |\cdot|_2$ are said to be *equivalent* if there exists a constant $C > 0$ such that

$$\frac{1}{C} \cdot |x|_2 \leq |x|_1 \leq C \cdot |x|_2$$

for all $x \in R$.

Example 2.2.

(1) If R is a subring of $\mathbb{C}$, the restriction of the absolute value turns R into a normed ring.

(2) Any ring can be made into a normed ring by equipping it with the *discrete norm*, given by
$$|x|_{\mathrm{d}} = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$

(3) Let Γ be a finitely generated group with word metric d_Γ and associated word length $\ell_\Gamma(\gamma) = d_\Gamma(\gamma, e)$. Let R be a normed ring. The associated *weighted norm* on the group ring $R\Gamma$ is defined as
$$\left\| \sum_{\gamma \in \Gamma} a_\gamma \gamma \right\|^\Gamma := \sum_{\gamma \in \Gamma} |a_\gamma| \, (1 + \ell_\Gamma(\gamma)).$$

It is a straightforward computation that this is a norm on $R\Gamma$.

For any normed ring R, there is an induced ℓ^1-*norm* on any free R-module with basis B given by
$$\left\| \sum_{b \in B} \alpha_b b \right\| = \sum_{b \in B} |\alpha_b|.$$
We can consider the group ring $R\Gamma$ as a free R-module with basis Γ and equip it with the ℓ^1-norm.

Let Γ be finitely generated. On free $R\Gamma$-modules L we consider two separate norms. The ℓ^1-norm $\|\cdot\|$ induced by viewing L as a free R-module and the weighted norm $\|\cdot\|^\Gamma$ coming from the weighted norm on $R\Gamma$.

If R is equipped with the discrete norm, we write $\|\cdot\|_{\mathrm{d}}$ and $\|\cdot\|_{\mathrm{d}}^\Gamma$ for the corresponding induced norms.

The following two properties of a ring are sometimes convenient to assume.

Definition 2.3. A norm on a ring is called *symmetric* if $|-x| = |x|$ for all $x \in R$. A norm is called ε-*separated* for $\varepsilon > 0$ if $|x| \geq \varepsilon$ for all $x \in R \setminus \{0\}$.

Examples of symmetric, ε-separated norms include the absolute value norm on $\mathbb{Z}$ and the discrete norm on any ring. For both we can choose $\varepsilon = 1$. In Section 4.4, we give an example showing that the assumption of an ε-separated norm is essential for $\mathrm{FV}_{R,\Gamma}^n$ to take finite values for all groups.

Lemma 2.4. *Every norm on a ring R is equivalent to a symmetric norm.*

Proof. One easily checks that $|r|' := |r| + |-r|$ defines a norm on R. By definition, this norm is symmetric and $|r| \leq |r|' \leq |r| + |-1| \, |r| = (1 + |-1|) \, |r|$. □

Lemma 2.5. *Let R be an ε-separated normed ring. Then up to equivalence of norms we may assume that R is 1-separated.*

Proof. Scaling a norm by a factor $C \geq 1$ defines a norm which is equivalent to the original one. Therefore, if $\varepsilon < 1$ in Definition 2.3, we can scale the norm by $\frac{1}{\varepsilon}$. □

The following lemma is proved in [BKV25, Lemma 2.3] for the non-weighted ℓ^1-norm on free modules. In the weighted case the proof is identical.

Lemma 2.6. *Let Γ be a finitely generated group and $f \colon M \to N$ be an $R\Gamma$-linear map between finitely generated free $R\Gamma$-modules. Then there exists a constant $C > 0$ such that $\|f(x)\|^\Gamma \leq C\|x\|^\Gamma$ for all $x \in M$.*

2.2. Filling functions.

Definition 2.7. Let $n \geq 0$. A group Γ is of type $\mathrm{FP}_n(R)$ if there exists a projective $R\Gamma$-resolution $P_* \to R$ of the trivial $R\Gamma$-module R, where P_i is finitely generated for $i \leq n$. By [Bro12, VIII 4.3], one may equivalently require the resolution to consist of free modules. If the resolution P_* can be chosen to be finite, that is, $P_i = 0$ for $i > n$ for some n, we say that Γ is of type $\mathrm{FP}(R)$. If n is minimal with this property $\mathrm{cd}_R(\Gamma) := n$ is called the *cohomological dimension* of Γ.

Remark 2.8. Let Γ be of type $\mathrm{FP}_n(R)$ with $n \geq 1$. Then there exists a free $R\Gamma$-resolution L_* of the trivial $R\Gamma$-module R such that L_i is finitely generated for $i \leq n$. We may assume that the modules L_1 and L_0 are the cellular chain modules of a Cayley graph of Γ. That is, if S is a finite generating set for Γ, then we may choose L_* to look like the following chain complex

$$\cdots \to L_2 \longrightarrow R\Gamma^{|S|} \xrightarrow{e_i \mapsto (s_i - 1)} R\Gamma \xrightarrow{\varepsilon} R.$$

Definition 2.9 ([KK21, Definition 2.1]). Let $n \geq 0$. We say that a projective $R\Gamma$-resolution $P_* \to R$ is *n-admissible* if P_n and P_{n+1} are finitely generated free $R\Gamma$-modules equipped with a choice of free bases.

The following definition is based on [BKV25, Definitions 2.9, 2.13].

Definition 2.10. Let R be a normed ring and Γ be a group of type $\mathrm{FP}_{n+1}(R)$. Let $P_* \to R$ be an n-admissible resolution. The *filling volume* of a cycle $c \in P_n$ is

$$\mathrm{FV}_R(c) := \inf_{\substack{b \in P_{n+1} \\ \partial b = c}} \|b\|.$$

The *weighted filling volume* of c is defined as

$$\mathrm{wFV}_R(c) := \inf_{\substack{b \in P_{n+1} \\ \partial b = c}} \|b\|^{\Gamma}.$$

The *nth homological filling function of Γ with coefficients in R* is the function $f_n^{R,\Gamma}(l) \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined through

$$f_n^{R,\Gamma}(l) := \sup_{\substack{c \in P_n \\ \partial c = 0, \|c\| \leq l}} \mathrm{FV}_R(c).$$

The weighted version is defined similarly by

$$\mathrm{w}f_n^{R,\Gamma}(l) := \sup_{\substack{c \in P_n \\ \partial c = 0, \|c\|^{\Gamma} \leq l}} \mathrm{wFV}_R(c).$$

Definition 2.11. Given functions $f, g \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we write $f \preceq g$ if there are constants $C, D > 0$ such that $f(v) \leq Cg(Cv + D) + Cv + D$ for every $v \geq 0$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$ hold.

The following lemma from [BKV25] tells us that up to $\approx$-equivalence $f_n^{R,\Gamma}$ and $\mathrm{w}f_n^{R,\Gamma}$ are well-defined notions. There it is proven for non-weighted filling functions, however the proof also applies if one replaces all norms by their weighted versions.

Lemma 2.12 ([BKV25, Lemma 2.12]). *Let P_* and P_*' be two n-admissible resolutions of the trivial $R\Gamma$-module R. Let f_n and f_n' denote the corresponding (weighted) filling functions from Definition 2.10, respectively. Then $f_n \approx f_n'$.*

Convention 2.13. For a ring R equipped with the discrete norm, the notation $\mathrm{DFV}^n_{R,\Gamma}$ is standard for $f^{R,\Gamma}_{n-1}$. In this case we also write $\mathrm{wDFV}^n_{R,\Gamma}$ for the weighted version. Note that there is an index shift between the two notations. In the following we use the index convention matching $\mathrm{DFV}^n_{R,\Gamma}$. To avoid confusion we write $\mathrm{FV}^n_{R,\Gamma}$ and $\mathrm{wFV}^n_{R,\Gamma}$ for $f^{R,\Gamma}_{n-1}$ and $\mathrm{w}f^{R,\Gamma}_{n-1}$ respectively.

Lemma 2.14. *Equivalent norms on R induce equivalent (weighted) filling functions.*

Proof. We show this for the non-weighted version. The proof is identical in the weighted case. Let $|\cdot|$, $|\cdot|'$ be equivalent norms and $C > 0$ such that $C^{-1}|x|' \leq |x| \leq C|x|'$ for all $x \in R$. We write R' for R equipped with the norm $|\cdot|'$ and $\|\cdot\|'$ for the ℓ^1-norm induced by $|\cdot|'$. Clearly $C^{-1} \cdot \|c\|' \leq \|c\| \leq C \cdot \|c\|'$ and $C^{-1} \cdot \mathrm{FV}_{R'}(c) \leq \mathrm{FV}_R(c) \leq C \cdot \mathrm{FV}_{R'}(c)$. Therefore,

$$\mathrm{FV}^n_{R,\Gamma}(l) = \sup\{\mathrm{FV}_R(c) \mid \|c\| \leq l\}$$
$$\leq \sup\{C \cdot \mathrm{FV}_{R'}(c) \mid \|c\|' \leq C \cdot l\} = C \cdot \mathrm{FV}^n_{R',\Gamma}(C \cdot l),$$

hence $\mathrm{FV}^n_{R,\Gamma} \preccurlyeq \mathrm{FV}^n_{R',\Gamma}$ and by symmetry $\mathrm{FV}^n_{R',\Gamma} \approx \mathrm{FV}^n_{R,\Gamma}$. $\square$

3. Cellular arguments in free chain-complexes

The goal of this chapter is to develop tools for translating geometric arguments in cell complexes to the setting of free chain complexes.

In the following, let Γ be a finitely generated group with a word metric d_Γ and associated word length $\ell_\Gamma(\gamma) = d_\Gamma(\gamma, e)$. Let $n \in \mathbb{N} \cup \{\infty\}$ and let L_* be a non-negative free $R\Gamma$-chain complex such that L_i is finitely generated for $i \leq n$. For each $i \leq n$ let B_i be a finite $R\Gamma$-basis of L_i. Then $\Gamma B_i = \{\gamma b \mid \gamma \in \Gamma,\, b \in B_i\}$ is an R-basis for L_i.

3.1. **A geometric structure on chain complexes.** Let $\alpha = \sum_{u \in \Gamma B_i} a_u u \in L_i$. The *$R$-support* of α is the set

$$\operatorname{supp}_R(\alpha) := \{u \in \Gamma B_i \mid a_u \neq 0\}.$$

Similarly, we define the *Γ-support* of α as

$$\operatorname{supp}_\Gamma(\alpha) := \{\gamma \in \Gamma \mid a_{\gamma b} \neq 0 \text{ for some } b \in B_i\}.$$

The functions d_Γ and ℓ_Γ can be extended to

$$d_\Gamma \colon \coprod_{i=0}^n \Gamma B_i \times \coprod_{i=0}^n \Gamma B_i \to \mathbb{N}_0, \quad \ell_\Gamma \colon \Gamma B_i \to \mathbb{N}_0$$

by setting $d_\Gamma(\gamma b, \gamma' b') := d_\Gamma(\gamma, \gamma')$ and $\ell_\Gamma(\gamma b) := \ell_\Gamma(\gamma)$. Note that d_Γ is only a pseudo-metric on $\coprod_i \Gamma B_i$.

Remark 3.1. Let $x = \gamma_x b \in \Gamma B_i$ and $y = \gamma_y b' \in \Gamma B_j$. Then

$$\ell_\Gamma(y) = \ell_\Gamma(\gamma_y) \leq \ell_\Gamma(\gamma_x) + d_\Gamma(\gamma_x, \gamma_Y) = \ell_\Gamma(x) + d_\Gamma(x, y),$$

hence $\|y\|^\Gamma = 1 + \ell_\Gamma(x) \leq d_\Gamma(x, y) = \|x\|^\Gamma + d_\Gamma(x, y)$.

Definition 3.2. Let U be a subset of the R-basis ΓB_i. We define a non-oriented graph $\mathrm{Gr}(U)$ with vertex set U and edges given by $\{u, v\}$ for $u \neq v \in U$ such that $\operatorname{supp}_R(\partial u) \cap \operatorname{supp}_R(\partial v) \neq \emptyset$. For an element $\alpha \in L_k$ we define $\mathrm{Gr}(\alpha)$ as $\mathrm{Gr}(\operatorname{supp}_R(\alpha))$. We say that U, respectively α, is *connected* if $\mathrm{Gr}(U)$, respectively $\mathrm{Gr}(\alpha)$, is connected.
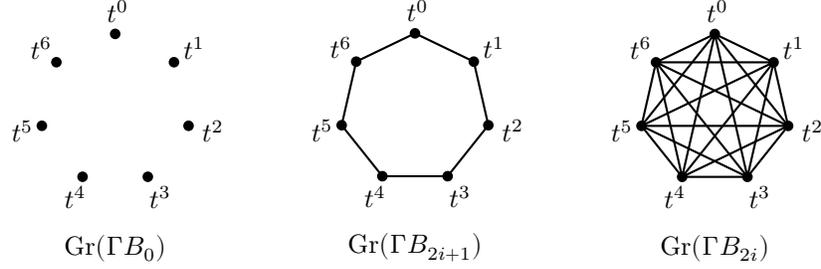
FIGURE 1. The graphs associated to the chain modules of the standard free $R\Gamma$-resolution of $\Gamma = \mathbb{Z}/7\mathbb{Z}$.

Remark 3.3. For $V \subset U$ the graph $\mathrm{Gr}(V)$ is the induced subgraph of $\mathrm{Gr}(U)$ on the vertices V. Therefore, if U is not connected, there exists a decomposition $U = \bigsqcup_j U_j$ such that each U_j is connected and of maximal size corresponding to a connected component of $\mathrm{Gr}(U)$. In particular, if $\alpha \in L_i$ is not connected, this decomposition of $\mathrm{supp}_R(\alpha)$ induces a decomposition $\alpha = \sum_{j=1}^m \alpha_j$ such that $\|\alpha\| = \sum_{j=1}^m \|\alpha_j\|$. If α is a cycle, then each α_i is also a cycle as no cancellation can happen between different connected components.

Example 3.4. If $\Gamma = \mathbb{Z}/k\mathbb{Z} = \langle t \mid t^k \rangle$ a free $R\Gamma$-resolution of R is given by

$$\cdots \xrightarrow{N\cdot} R\Gamma \xrightarrow{(1-t)\cdot} R\Gamma \xrightarrow{N\cdot} R\Gamma \xrightarrow{(1-t)\cdot} R\Gamma \xrightarrow{\varepsilon} R,$$

where $N = \sum_{i=0}^{k-1} t^i$. For each i a free R-basis of $R\Gamma$ is $\Gamma B_i = \{t^j \mid 0 \leq j \leq k-1\}$. We have $N \cdot t^j = N$ and $(1-t) \cdot t^j = t^j - t^{j+1}$. Therefore, $\mathrm{Gr}(\Gamma B_n)$ is a cycle on k vertices if n is odd and a complete graph if n is even, except for $n = 0$, as $\mathrm{Gr}(\Gamma B_0)$ does not contain any edges (see Fig. 1).

Assume that L_* is the cellular chain complex of a free simplicial Γ-complex X. Then ΓB_i corresponds bijectively to the set of i-cells of X, and $\mathrm{Gr}(\Gamma B_i)$ is the graph whose edges connect cells sharing a common face. We endow X with the metric induced by the Euclidean metric on each simplex, which in turn defines a metric on the sets of simplices via the distance between their barycentres. This allows us to bound the distance between neighbouring cells. The next lemma provides an algebraic analogue of this phenomenon.

Lemma 3.5. *Let $0 \leq i \leq n$ and $U \subseteq \Gamma B_i$. There exists $K_i \geq 0$ such that*

(1) *if $x \in U$ and $v \in \mathrm{supp}_R(\partial x)$, then $d_\Gamma(x, v) \leq K_i$ and $\|x\|^\Gamma \leq \|v\|^\Gamma + K_i$;*
(2) *if $x, y \in U$ are connected by an edge in $\mathrm{Gr}(U)$, then $d_\Gamma(x, y) \leq 2K_i$ and $\|y\|^\Gamma \leq \|x\|^\Gamma + 2 \cdot K_i$.*

Proof. In both cases we just show the estimate on the distance, as the statement for the norm follows from Remark 3.1. Consider the set $A_i := \bigcup_{b^{(i)} \in B_i} \mathrm{supp}_\Gamma(\partial b^{(i)})$. If A_i is empty then either B_i is empty or $\partial_i = 0$. In either case both statements of the lemma are vacuously true, so we may set $K_i = 0$. From now on let us assume that A_i is non-empty and define $K_i := \max\{\ell_\Gamma(h) \mid h \in A_i\} < \infty$.

Let $x = \gamma_x b_x^{(i)} \in U$ and $v = \gamma_v b_v^{(i-1)} \in \mathrm{supp}_R(\partial x)$. Since $\partial x = \gamma_x \partial b_x^{(i)}$, we can write $\gamma_v = \gamma_x h$ for some $h \in A_i(b_x^{(i)})$. It follows that $d_\Gamma(x, v) = \ell_\Gamma(h) \leq K_i$.

Now let $y = \gamma_y b_y^{(i)}$ be another element in U such that x and y are connected by an edge in $\mathrm{Gr}(U)$. Then there exist $h_x, h_y \in A_i$ and an $R\Gamma$-basis element $b^{(i-1)} \in B_{i-1}$ such that
$$\gamma_x h_x b^{(i-1)} = \gamma_y h_y b^{(i-1)} \in \mathrm{supp}_R(\partial x) \cap \mathrm{supp}_R(\partial y).$$
Hence, $\gamma_y = \gamma_x h_x h_y^{-1}$ and therefore $d_\Gamma(x, y) \leq \ell_\Gamma(h_x) + \ell_\Gamma(h_y) \leq 2K_i$. $\qquad\square$

Corollary 3.6. *Let $0 \leq i \leq n$, $c \in L_i$ be connected, and let $x, y \in \mathrm{supp}_R(c)$.*

(1) $d_\Gamma(x, y) \leq \left|\mathrm{supp}_R(c)\right| \cdot 2K_i$ *and* $\|y\|^\Gamma \leq \|x\|^\Gamma + \left|\mathrm{supp}_R(c)\right| \cdot 2K_i$;

Further assume that R is 1-separated.

(2) $\|y\|^\Gamma \leq \|x\|^\Gamma + 2K_i\|c\|$ *and* $\|c\|^\Gamma \leq (\|x\|^\Gamma + 2K_i\|c\|)\|c\|$.

Proof. The diameter of $\mathrm{Gr}(c)$ is at most $\left|\mathrm{supp}_R(c)\right|$, hence the first claim follows from Lemma 3.5. If R is 1-separated, $\|c\| \geq \left|\mathrm{supp}_R(c)\right|$. Writing $c = \sum_{\gamma b \in \Gamma B_i} \alpha_{\gamma b} \gamma b$ we obtain
$$\|c\|^\Gamma = \sum_{\gamma b} |\alpha_{\gamma b}| \, \|\gamma b\|^\Gamma \leq \sum_{\gamma b} |\alpha_{\gamma b}| \left(\|x\|^\Gamma + \left|\mathrm{supp}_R(c)\right| 2K_i\right)$$
$$\leq (\|x\|^\Gamma + 2K_i\|c\|)\|c\|. \qquad\square$$

Definition 3.7. Let $\alpha \in L_i$. We say α is *reduced* if there does not exist a decomposition $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \neq 0$ such that $\partial \alpha_1 = 0$.

Lemma 3.8. *There exists a constant $C_i \geq 0$ such that for every reduced chain $\alpha \in L_i$ and every $u \in \mathrm{supp}_R(\alpha)$ there is $v_u \in \mathrm{supp}_R(\partial \alpha)$ with $d_\Gamma(u, v_u) \leq C_i \left|\mathrm{supp}_R(\alpha)\right|$ and thus $\|u\|^\Gamma \leq \|v_u\|^\Gamma + C_i\|\alpha\|_\mathrm{d}$.*

Proof. Decompose α into its connected components $\alpha = \alpha_1 + \cdots + \alpha_k$. Because α is reduced, $\partial \alpha_p \neq 0$ for every $1 \leq p \leq n$. Hence, we can pick vertices u_p of $\mathrm{Gr}(\alpha_p)$ such that $\mathrm{supp}_R(\partial u_p) \cap \mathrm{supp}_R(\partial \alpha_p) \neq \emptyset$. Let $v_p \in \mathrm{supp}_R(\partial u_p) \cap \mathrm{supp}_R(\partial \alpha_p)$ be such an element in the intersection. By Lemma 3.5, $\|u_p\|^\Gamma \leq \|v_p\|^\Gamma + K_i$ and by Corollary 3.6,
$$\|u\|^\Gamma \leq \|u_p\|^\Gamma + \left|\mathrm{supp}_R(\alpha_p)\right| \cdot 2K_i \leq \|v_p\|^\Gamma + K_i + \left|\mathrm{supp}_R(\alpha)\right| \cdot 2K_i$$
for each $u \in \mathrm{supp}_R(\alpha_p)$. A suitable constant is therefore given by $C_i := 3K_i$. $\qquad\square$

Lemma 3.9. $\mathrm{Gr}(\Gamma B_i)$ *is locally finite for all $0 \leq i \leq n$.*

Proof. Because the differentials of L_* are Γ-equivariant, Γ acts on $\mathrm{Gr}(\Gamma B_i)$ by graph automorphisms. It therefore suffices to show that
$$B_{\mathrm{Gr}(\Gamma B_i)}(b, 1) = \{\gamma b' \in \Gamma B_i \mid \{\gamma b', b\} \text{ is an edge in } \mathrm{Gr}(\Gamma B_i)\}$$
is finite for every $b \in B_i$. To this end let $\gamma b' \in \Gamma B_i$ be connected to b by an edge in $\mathrm{Gr}(\Gamma B_i)$. Then by Lemma 3.5 we have that
$$\ell_\Gamma(\gamma) = \ell_\Gamma(\gamma b') \leq \ell_\Gamma(b) + 2K_i = 2K_i$$
and hence $\ell_\Gamma(\gamma) \leq 2K_i$. As Γ is finitely generated and B_i is finite, we conclude that $B_{\mathrm{Gr}(\Gamma B_i)}(b, 1)$ is finite. $\qquad\square$

For $i = 0$ the above statement is trivially satisfied, as B_{-1} is empty and therefore $\mathrm{Gr}(\Gamma B_0)$ contains no edges at all.

Lemma 3.10. *Let $0 \leq i \leq n-1$ and $u \in \Gamma B_i$. Then the set*

$$S(u) := \{v \in \Gamma B_{i+1} \mid u \in \mathrm{supp}_R(\partial v)\}$$

is finite.

Proof. If $S(u)$ is empty or consists of only one element, there is nothing to show. Otherwise, fix some $v_0 \in S(u)$. For any $w \in S(u)$ such that $w \neq v_0$, the graph $\mathrm{Gr}(\Gamma B_{i+1})$ has $\{v_0, w\}$ as an edge. Therefore, $S(u)$ is contained in $B_{\mathrm{Gr}(\Gamma B_{i+1})}(v_0, 1)$ which is finite by Lemma 3.9. $\square$

3.2. **Non-equivariant subcomplexes.** We now study algebraic analogues of subcomplexes of a simplicial Γ-complex. For $l > 0$ consider the collection

$$\mathcal{C}_l^i := \{U \subset \Gamma B_i \mid |U| \leq l, U \text{ connected}\}.$$

An element $U \in \mathcal{C}_l^i$ can be thought of as giving a basis to a connected R-subspaces of L_i with R-dimension at most l. The group Γ acts on $\mathcal{C}_l^i$ by left multiplication, that is, $\gamma U := \{\gamma u \mid u \in U\}$.

Proposition 3.11. *Let $l \geq 0$ and $0 \leq i \leq n$. Then $\Gamma \backslash \mathcal{C}_l^i$ is finite.*

Proof. Let $U \in \mathcal{C}_l^i$. If $U = \emptyset$, the orbit ΓU just consists of U. If $U \neq \emptyset$ let $\gamma b \in U$. By considering $U' := \gamma^{-1} U$, we may assume that $\gamma = 1$ and $b \in U$. Because $|U| \leq l$, the diameter of $\mathrm{Gr}(U)$ is at most l. Repeated application of Lemma 3.5 yields that for any $\gamma' b' \in U$ we have

$$\ell_\Gamma(\gamma') = \ell_\Gamma(\gamma' b') \leq \ell_\Gamma(b) + 2K_i l = 2K_i l.$$

Therefore, U is a subset of the finite set $\{\gamma \in \Gamma \mid l_\Gamma(\gamma) \leq 2K_i l\} B_i$. $\square$

Definition 3.12. A *basis collection* is a collection U_* of subsets $U_i \subseteq \Gamma B_i$ for $i \geq 0$. We say a basis collection U_* is *closed under differentials* if $\mathrm{supp}_R(\partial u) \subseteq U_{i-1}$ for all $u \in U_i$ and all $i > 0$. If U_* is closed under differentials, the modules $L_i(U_*) := \langle U_i \rangle_R$ assemble to an R-subcomplex $L_*(U_*)$ of L_*.

A basis collection U_* is called *at most k-dimensional* if $U_i = \emptyset$ for all $i > k$ and *k-dimensional* if additionally $U_k \neq \emptyset$. It is said to be of *finite type* if U_i is finite for all $i \geq 0$.

Remark 3.13. We may understand a single subset $U \subseteq \Gamma B_k$ as a k-dimensional basis collection by choosing the empty subset of ΓB_j for $j \neq k$. By abuse of notation, we also denote this collection by U.

Let $j \geq 0$ and $U_j \subseteq \Gamma B_j$ be some subset. We inductively define

$$P_i^j(U_j) := \begin{cases} 0 & \text{if } i > j; \\ U_j & \text{if } i = j; \\ \bigcup_{u \in P_{i+1}^j(U)} \mathrm{supp}_R(\partial u) & \text{if } i < j, \end{cases}$$

which is closed under differentials. More generally, if we are given a collection U_* of subsets $U_i \subseteq \Gamma B_i$ for all $i \geq 0$, we define $P_i(U_*) := \bigcup_{j \geq 0} P_i^j(U_j)$. If U_* is already closed under differentials then $P_*(U_*) = U_*$.

If L_* is the cellular chain complex of some simplicial complex X a basis collection U_* corresponds to a choice of subsets of i-cells of X for each $i > 0$. Then $L_*(P_*(U_*))$ is the cellular chain complex of the subcomplex generated by these cells.
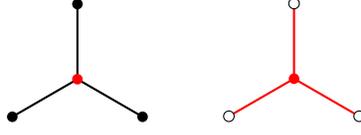
FIGURE 2.   On the left a tripod with subcomplex Y given by the central vertex. On the right the set $\widetilde{N}_*(Y)$.

3.3. **Thickening of subcomplexes.** Next we describe a procedure which can be understood as an algebraic analogue of the following geometric construction. Given a k-dimensional subcomplex Y of a simplicial complex X, we define $N(Y)$ to be the subcomplex consisting of Y together with all cells that have a boundary face in Y.

Let K_* be an R-subcomplex of L_* equipped with a basis $U_i \subseteq \Gamma B_i$ for all i. First we consider the sets

$$\widetilde{N}_i(U_*) := \begin{cases} U_0 & \text{if } i = 0; \\ U_i \cup \bigcup_{u \in U_{i-1}} S(u) & \text{if } i > 0, \end{cases}$$

where $S(u)$ is as in Lemma 3.10. Note that if U_* is at most k-dimensional, $\widetilde{N}_i(U)$ is empty for $i > k + 1$.

On the level of simplicial complexes $\widetilde{N}_i(U_*)$ corresponds to the collection of i-cells intersecting the subcomplex Y. In general this collection is not closed under differentials. For example let X be a simplicial tripod and Y be the central vertex. Then $\widetilde{N}_0 = Y_0$ and $\widetilde{N}_1$ is the collection of all three edges. However, these sets do not form a simplicial complex because the outer vertices are missing (see Fig. 2). Applying $P_i(-)$ resolves this deficit.

For a chain complex C_* we write $C_*^{(k)}$ for its k-skeleton, that is, $C_i^{(k)}$ is C_i for $i \le k$ and 0 for $i > k$. For a basis collection U_*, we write $U_{* \le k}$ for the truncated collection with the empty set in degrees $i > k$. Note that $L_*(U_{\le k}) = L_*(U_*)^{(k)}$.

Definition 3.14. Let U_* be a basis collection. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $j \ge 0$ we inductively define the *j-step k-thickening* of U as the basis collection

$$N_i^{k,j}(U_*) := \begin{cases} P_*(U_*) & \text{if } j = 0; \\ P_i(\widetilde{N}_{* \le k}(P_*(U_*))) = \bigcup_{l=0}^{k} P_i^l(\widetilde{N}_l(P_*(U_*))) & \text{if } j = 1; \\ N_*^{k,1}(N_*^{k,j-1}(U_*)) & \text{if } j > 1. \end{cases}$$

The associated R-subcomplexes are denoted by $N^{k,j} L_*(U)$.

Lemma 3.15. *Let U_* be a basis collection. Then*

(1) $\widetilde{N}_i(U_*)$ *is finite if either*
   (i) $i = 0$ *and U_0 is finite, or*
   (ii) $1 \le i \le n$ *and both U_i and U_{i-1} are finite;*
(2) $P_i(U_*)$ *is finite if U_j is finite for all $j \ge i$.*

Proof. The first implication follows directly from Lemma 3.10. For the second statement note that $P_i(U_*)$ is finite if $P_i^j(U_j)$ is finite for all $j \ge 0$. However, $P_i^j(U_j)$ is empty for $j < i$. If $j \ge i$ then by construction $P_i^j(U_j)$ is finite if and only if U_j is finite. □

An immediate consequence is the following lemma.

Lemma 3.16. *Let $0 \leq k \leq n$ and U_* be an at most k-dimensional basis collection of finite type. Then $N_i^{k,1}(U_*)$ is at most k-dimensional and of finite type.*

Proposition 3.17. *Let $0 \leq k \leq n$. Let U_* be an at most k-dimensional basis collection of finite type. Then $N_*^{k,j}(U_*)$ is at most k-dimensional and of finite type for all $j \geq 0$.*

Proof. By definition, $N_*^{k,j}(U_*)$ is always at most k-dimensional. Therefore, the claim follows from Lemma 3.16 by induction on j. $\qquad\square$

Theorem 3.18. *Let $1 \leq m \leq k \leq n$ and $U \in \mathcal{C}_l^m$. Assume that*

  (i) *U is non-empty;*
  (ii) *$\partial b \neq 0$ for all $b \in B_i$ and $1 \leq i \leq k$;*
  (iii) *$\mathrm{Gr}(\Gamma B_1)$ is connected;*
  (iv) *$\bigcup_{u \in \Gamma B_1} \mathrm{supp}_R(\partial u) = \Gamma B_0$.*

Then $\mathrm{colim}_j N^{k,j} L_*(U)^{(k)} = L_*^{(k)}$.

Proof. We inductively show that $\bigcup_j N_i^{k,j}(U) = \Gamma B_i$ for all $i \leq k$. The case $i = 0$ follows directly from condition (iv). Note that $N_1^{k,0}(U) = P_1(U)$. From conditions (i) and (ii) it follows that $P_1(U) \neq \emptyset$. Let $u \in P_1(U)$ and $v \in \Gamma B_1$. By condition (iii), there is some path $p_{u,v}$ in $\mathrm{Gr}(\Gamma B_1)$ from u to v. Then $v \in N_1^{k,l}(U)$, where l is the length of the path $p_{u,v}$.

Now let $1 < i \leq k$ and assume $\bigcup_j N_{i-1}^{k,j}(U) = \Gamma B_{i-1}$. For any $u \in \Gamma B_i$ condition (ii) guarantees that $\mathrm{supp}_R(\partial u)$ is non-empty. Choose j large enough such that $\mathrm{supp}_R(\partial u) \cap N_{i-1}^{k,j}(U) \neq \emptyset$, so $u \in N_i^{k,j+1}(U)$. Hence, $\bigcup_j N_i^{k,j}(U) = \Gamma B_i$. $\qquad\square$

4. Finiteness of filling functions

In this section we prove finiteness for various filling functions introduced in Section 2.2.

4.1. **Integral filling functions.** The following proposition recovers the result that $\mathrm{FV}_{\mathbb{Z},\Gamma}^n$ takes finite values [FMP18].

Proposition 4.1. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Assume that for each $l \geq 0$ the set $R_{\leq l} := \{r \in R \mid |r| \leq l\}$ is finite. Then $\mathrm{FV}_{R,\Gamma}^n$ takes finite values.*

Proof. We consider the collection $\mathcal{D}_l^{n-1} := \{c \in L_{n-1} \mid c \text{ connected}, \|c\| \leq l\}$ of connected $(n-1)$-cycles with norm at most l. The group Γ acts on $\mathcal{D}_l^{n-1}$ by left multiplication, as $\|\gamma c\| = \|c\| \leq l$ for each $c \in \mathcal{D}_l^{n-1}$ and $\gamma \in \Gamma$. By assumption, $R_{\leq 1}$ is finite, hence R is ε-separate for some $\varepsilon > 0$. Using Lemma 2.5 we may assume that $\varepsilon = 1$, hence $\left|\mathrm{supp}_R(c)\right| \leq \|c\| \leq l$. Let $U_1, \dots, U_N$ be representatives of $\Gamma \backslash \mathcal{C}_l^{n-1}$. If $c \in \mathcal{D}_l^{n-1}$ is a cycle with $\mathrm{supp}_R(c) = U_j$, it is contained in the set

$$R_{\leq l} U_j := \Big\{ \sum_{u \in U_j} \alpha_u u \mid \alpha_u \in R_{\leq l} \Big\},$$

which is finite as $R_{\leq l}$ and U_j are finite. We conclude that $\Gamma \backslash \mathcal{D}_l^{n-1}$ is finite. This implies that there is some $D(l) \geq 0$ such that $\mathrm{FV}_R(c) \leq D(l)$ for every $c \in \mathcal{D}_l^{n-1}$. In the general case a cycle c decomposes as a sum $c = c_1 + \cdots + c_s$ of connected cycles.

Then $\|c\| = \sum_i \|c_i\|$ and each c_i has a filling b_i satisfying $\|b_i\| \leq D(\|c_i\|)$. Therefore, $\sum_i b_i$ is a filling for c and we conclude

$$\mathrm{FV}^n_{R,\Gamma}(l) \leq \max\Big\{\sum_{i=1}^m D(l_i) \mid \sum_{i=1}^m l_i = l, l_i \in \mathbb{N}\Big\} < \infty. \qquad \square$$

Corollary 4.2. $\mathrm{FV}^n_{\mathbb{Z},\Gamma}$ *takes finite values.*

4.2. **Discrete filling functions.** To show that $\mathrm{DFV}^n_{R,\Gamma}(l)$ is finite we follow a similar approach as [BKV25, Proposition 2.21]. There are only finitely many orbits for $\mathrm{supp}_R(c)$ of a connected cycle c with $\|c\|_{\mathrm{d}} \leq l$. Therefore, we can use the thickening procedure from the previous section to obtain a subcomplex of finite R-dimension containing a filling for all connected cycles. An upper bound on $\|b\|_{\mathrm{d}}$ for a filling b of c is given by the dimension of this subcomplex in degree n. To apply Theorem 3.18 we first verify that there exists a suitable resolution satisfying the assumptions of the theorem.

Proposition 4.3. *Let $n \geq 1$ and Γ be of type $\mathrm{FP}_n(R)$. Then there exists a free $R\Gamma$-resolution $L_* \to R$ such that L_i is finitely generated for $i \leq n$ and a choice of bases B_i of L_i such that conditions* (ii) *to* (iv) *from Theorem 3.18 are satisfied.*

Proof. Let S be a finite generating set for Γ. As in Remark 2.8, we start off our resolution with the cellular chain complex $L_i := C_i(\mathrm{Cay}(\Gamma, S); R)$ for $i = 0, 1$. Then $\mathrm{Gr}(\Gamma B_1)$ is the line graph of a Cayley graph of Γ.[1] In particular $L(G)$ is connected if G is connected. Therefore, L_* satisfies conditions (iii) and (iv).

For condition (ii) note that we can extend our partial resolution to a full resolution by iteratively choosing surjections $\partial_i \colon L_i \twoheadrightarrow \ker(L_i \to L_{i-1})$ with L_i of minimal $R\Gamma$-rank. If a basis element of L_i were to be mapped to zero under ∂_i, this would violate minimality. The fact that we can choose L_i to be finitely generated for $i \leq n$ follows from Schanuel's Lemma [Bro12, VIII 4.3]. $\square$

Theorem 4.4. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Then $\mathrm{DFV}^n_{R,\Gamma}$ takes finite values.*

Proof. We take L_* to be the free resolution from Proposition 4.3. Let $l > 0$ and let $U_1, \ldots, U_m$ be a choice of representatives of the Γ-orbits in $\mathcal{C}^{n-1}_l$. Because $H_{n-1}(L_*) = 0$, the inclusion $N^0 L_*(U_i) \to L_*$ induces the trivial map in $(n-1)$th homology. Since homology commutes with direct limits and $H_{n-1}(N^0 L_*(U_i))$ is a finitely generated R-module, it follows from Theorem 3.18 that there is an a_i such that the inclusion $\iota_{a_i} \colon N^0 L_*(U_i) \to N^{n,a_i} L_*(U_i)$ induces the trivial map

$$H_{n-1}(\iota_{a_i}) \colon H_{n-1}(N^{n,0} L_*(U_i)) \to H_{n-1}(N^{n,a_i} L_*(U_i)).$$

Define M^i_* to be $N^{n,a_i} L_*(U_i)$.

Let z be a connected $(n-1)$-cycle in L_{n-1} with $\|z\|_{\mathrm{d}} \leq l$. Then there exists $1 \leq i \leq m$ and $\gamma \in \Gamma$ such that $\mathrm{supp}_R(z) = \gamma U_i$. As $\gamma^{-1}z$ and z have the same filling volume, we may assume that $\gamma = 1$. As $H_{n-1}(\iota_{a_i}) = 0$, there exists a filling $b \in M^i_n$. The norm of b is bounded by $\big|N^{n,a_i}_n(U_i)\big| =: D_i(l)$. By Proposition 3.17, $D_i(l) < \infty$. Therefore, any cycle with connected support satisfying $\|z\|_{\mathrm{d}} \leq l$ has a filling bounded by $D(l) = \max\{D_i(l) \mid 1 \leq i \leq m\} < \infty$.

In the non-connected case one argues as in the proof of Proposition 4.1. $\square$

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[1]The line graph of a graph G is the graph $L(G)$ with the edges of G as vertices and edges given by the intersection of edges in G.

4.3. **Weighted filling functions.** In the following let Γ be a group of type $\mathrm{FP}_n(R)$. We fix a free resolution $L_* \to R$ and finite $R[\Gamma]$-bases of L_i for $i \leq n$.

Lemma 4.5. *Let $2 \leq i \leq n$. Assume that R is 1-separated and $\mathrm{wFV}_{R,\Gamma}^i(l) < \infty$ for all l. For any $(i-1)$-cycle $c \in L_{i-1}$ there exists a reduced filling y such that*

$$\|y\|^\Gamma \leq \|c\|\|c\|^\Gamma A_\Gamma^i(\|c\|) \quad and \quad \|y\| \leq \|c\|A_\Gamma^i(\|c\|),$$

where K_{i-1} is the constant from Lemma 3.5 and

$$A_\Gamma^i(x) = \mathrm{wFV}_{R,\Gamma}^i(x + 2K_{i-1}x^2) + 1.$$

Proof. Because R is 1-separated, c admits a decomposition $c = c_1 + \cdots + c_m$ into connected cycles with $m \leq \|c\|$. For $h_j \in \mathrm{supp}_\Gamma(c_j)$ consider the cycles $c_j' := h_j^{-1}c_j$. Choose minimal fillings y_j' of c_j' such that

$$\left\|y_j'\right\|^\Gamma \leq \mathrm{wFV}_{R,\Gamma}^i(\|c_j'\|^\Gamma) + 1$$

and let $y_j := h_j y_j'$. The $+1$-term is needed as we cannot assume that the minimum in the definition of FV_R is attained. Then $\partial y_j = h_j \partial y_j' = h_j h_j^{-1} c_j = c_j$, hence $y := y_1 + \cdots + y_m$ is a filling of c. We may assume that y is reduced, as otherwise we can remove the non-reduced part to obtain a smaller filling. Because $h_j \in \mathrm{supp}_\Gamma(c_j)$, there is some $v \in \mathrm{supp}_R(c_j)$ such that $v = h_j b_j$ for some $R\Gamma$-basis element b_j. Then $b_j \in \mathrm{supp}_R(c_j')$, hence by Corollary 3.6

$$\left\|c_j'\right\|^\Gamma \leq (\|b_j\|^\Gamma + 2K_{i-1}\|c_j\|)\|c_j\| = (1 + 2K_{i-1}\|c_j\|)\|c_j\| \leq (1 + 2K_{i-1}\|c\|)\|c\|.$$

It follows that

$$\left\|y_j'\right\|^\Gamma \leq \mathrm{wFV}_{R,\Gamma}^i(\|c_j'\|^\Gamma) + 1 \leq \mathrm{wFV}_{R,\Gamma}^i((1 + 2K_{i-1}\|c\|)\|c\|) + 1 = A_\Gamma^i(\|c\|).$$

Because R is 1-separated, we have $1 + \ell_\Gamma(h_j) \leq \|c_j\|^\Gamma \leq \|c\|^\Gamma$. Hence,

$$\left\|y_j\right\|^\Gamma \leq (1 + \ell_\Gamma(h_j))\|y_j'\|^\Gamma \leq \|c\|^\Gamma A_\Gamma^i(\|c\|).$$

In total, we obtain that

$$\|y\|^\Gamma \leq \sum_{j=1}^m \left\|y_j\right\|^\Gamma \leq \|c\|\|c\|^\Gamma A_\Gamma^i(\|c\|)$$

and

$$\|y\| \leq \sum_{j=1}^m \|y_j\| = \sum_{j=1}^m \|y_j'\| \leq \sum_{j=1}^m \|y_j'\|^\Gamma \leq \|c\|A_\Gamma^i(\|c\|). \qquad \square$$

We now give a criterion for the finiteness of weighted filling functions in terms of the finiteness of the non-weighted functions. The proof is based on the constructions in the proofs of [JR09, Corollary 2.5] and [BS23, Lemma 6.7].

Proposition 4.6. *Assume that R is 1-separated. For $2 \leq i \leq n$ there are constants $A_i, B_i > 0$ such that*

(1) $\mathrm{FV}_{R,\Gamma}^i(x) \leq x \cdot \mathrm{wFV}_{R,\Gamma}^i(x + A_i \cdot x^2) + 1$,

(2) $\mathrm{wFV}_{R,\Gamma}^i(x) \leq B_i \cdot \left(\mathrm{FV}_{R,\Gamma}^i(x) + 1\right)^2 + x \cdot \left(\mathrm{FV}_{R,\Gamma}^i(x) + 1\right)$.

In particular $\mathrm{FV}_{R,\Gamma}^i$ is finite if and only if $\mathrm{wFV}_{R,\Gamma}^i$ is finite.

Proof. Let $f(x) = \mathrm{FV}^i_{R,\Gamma}(x)+1$, $c \in L_{i-1}$ be an $(i-1)$-cycle, and let $b = \sum_u b_u u \in L_i$ be a reduced filling of c such that $\|b\| \leq f(\|c\|)$. By Lemma 3.8, there exist $v_u \in \mathrm{supp}_R(c)$ such that

$$\|u\|^{\Gamma} \leq \|v_u\|^{\Gamma} + C_i\|b\|_{\mathrm{d}} \leq \|c\|^{\Gamma} + C_i\|b\|,$$

where in the second inequality we use that R is 1-separated. Hence,

$$\begin{aligned}
\|b\|^{\Gamma} = \sum_{u \in \mathrm{supp}_R(b)} |b_u|\,\|u\|^{\Gamma} &\leq \|b\|(\|c\|^{\Gamma} + C_i\|b\|) \\
&\leq f(\|c\|)\|c\|^{\Gamma} + C_i f(\|c\|)^2 \\
&\leq f(\|c\|^{\Gamma})\|c\|^{\Gamma} + C_i f(\|c\|^{\Gamma})^2.
\end{aligned}$$

We conclude that $\mathrm{wFV}^i_{R,\Gamma}(x) \leq C_i f(x)^2 + x \cdot f(x)$.

Let $g(x) = \mathrm{wFV}^i_{R,\Gamma}(x)$ and $c \in L_{i-1}$ be a cycle. We first assume that $i \geq 2$ and consider the filling y of c from Lemma 4.5. It satisfies $\|y\| \leq \|c\|g(\|c\|+2K_{i-1}\|c\|^2)+1$ and thus,

$$\mathrm{FV}^i_{R,\Gamma}(x) \leq x \cdot g(x + 2K_{i-1}x^2) + 1. \qquad \square$$

Remark 4.7. For $n = 1$, the function $\mathrm{FV}^1_{R,\Gamma}$ is finite if and only if Γ is finite. This is because $\mathrm{FV}_R(\gamma - 1)$ grows like $\ell_\Gamma(\gamma)$ for $\gamma \in \Gamma$. In contrast, $\mathrm{wFV}^1_{R,\Gamma}$ is always linear, as $\|\gamma - 1\|^{\Gamma}$ also grows linearly in $\ell_\Gamma(\gamma)$.

Corollary 4.8. $\mathrm{wDFV}^n_{R,\Gamma}$ *takes finite values.*

4.4. **A non-finite example.** In [BKV25], the authors ask the following question about the finiteness of filling functions.

Question ([BKV25, 2.10])**.** For which norms $|\cdot|$ does the nth homological filling function take finite values?

We conclude this section with a short example, answering this question in the negative for all normed rings which are not ε-separated. This justifies why ε-separation is a natural condition in the context of filling functions if one wants $\mathrm{FV}^n_{R,\Gamma}$ to yield a finite invariant.

Example 4.9. Assume that R is not ε-separated for every $\varepsilon > 0$. Then there exists a strictly increasing sequence $n_i \in \mathbb{N}$ and elements $r_i \in R$ such that $\frac{1}{n_i+1} < |r_i| \leq \frac{1}{n_i}$. By Lemmas 2.4 and 2.14, we may assume that $|r| = |-r|$ for all $r \in R$.

Let $\Gamma = \mathbb{Z}^2 = \langle x, y \mid [x,y] \rangle$ and X be the universal cover of its presentation complex, that is, $X = \mathbb{R}^2$ with $X^{(1)} = \mathbb{Z}^2$. Consider the 1-cycles $\alpha_i = r_i \cdot [x^{n_i}, y^{n_i}]$ in X. They satisfy $\|\alpha_i\| = |r_i| \cdot 4n_i \leq 4$ and $\mathrm{FV}_R(\alpha_i) = |r_i| \cdot n_i^2 > \frac{n_i^2}{n_i+1} > \frac{n_i}{2}$. In particular $\mathrm{FV}^2_{R,\mathbb{Z}^2}(4) \geq \frac{n_i}{2}$ for all i, so $\mathrm{FV}^2_{R,\mathbb{Z}^2}$ is not finite.

However, under the assumption that R is ε-separated one could expect a positive answer to the above question.

5. QUASI-ISOMETRY INVARIANCE OF FILLING FUNCTIONS

The goal of this section is to prove quasi-isometric invariance of $\mathrm{FV}^n_{R,\Gamma}$ and $\mathrm{wFV}^n_{R,\Gamma}$ for groups of type $\mathrm{FP}_n(R)$ under the assumption that the homological filling functions of Γ take finite values. The proofs in this section build upon the proofs in [BKV25, Section 2].

In the following, let G and H be finitely generated groups of type $\mathrm{FP}_n(R)$. Let S and T be finite generating sets for G and H, respectively. Furthermore, let L_*^G and L_*^H be free resolutions as described in Remark 2.8.

Lemma 5.1. *Let R be 1-separated. Assume that $\mathrm{wFV}_{R,H}^i(l) < \infty$ for all $2 \le i \le n$ and all l. Let $f \colon G \to H$ be a (K,K)-quasi-isometry. Then for $i \le n$ there exist R-linear maps $f_i \colon L_i^G \to L_i^H$ and constants D_i, $\widetilde{D}_i$ such that*

(1) *f_0 is the R-linear extension of f;*
(2) *$f_{i-1}\partial = \partial f_i$ for $1 \le i \le n$;*
(3) *$\|f_i(x)\| \le D_i\|x\|$ for all $x \in L_i^G$ and $i \le n$;*
(4) *$\|f_i(x)\|^H \le \widetilde{D}_i\|x\|^G$ for all $x \in L_i^G$ and $i \le n$.*

Proof. Let $f_0 \colon RG \to RH$ be the linear extension of f. Then $\|f_0(g)\| = \|g\|$ for $g \in G$ and

$$\|f_0(g)\|^H = 1 + \ell_H(f(g)) \le 1 + K\ell_G(g) + K \le (1+K)\|g\|^G.$$

We set $D_0 := 1$ and $\widetilde{D}_0 := 2K$.

An R-basis of $L_1^G = RG^{|S|}$ is given by $\{gb_s \mid g \in G, s \in S\}$. For such a basis element gb_s we have $\partial gb_s = g(s-1) = gs - g$. Let $\gamma_{g,s}$ be a path from $f_0(g)$ to $f_0(gs)$ in $\mathrm{Cay}(H,T)$. Since f is a (K,K)-quasi-isometry, we can arrange that $\gamma_{g,s}$ has length at most $2K + 1$. Because we have assumed that the resolution L_*^H coincides with $C_i(\mathrm{Cay}(H,S); R)$ for $i = 0,1$, we may consider $\gamma_{g,s}$ as an element of $L_1^H = RH^{|T|}$ and define $f_1(gb_s) := \gamma_{g,s}$. Then $\gamma_{g,s}$ satisfies $\|\gamma_{g,s}\| \le (2K+1)|1_R|$ and we set $D_1 := 2K + 1$. Write $\gamma_{g,s} = \sum_{j=0}^m e_j$ with $m \le 2K$. It follows from Lemma 3.5 that $\|e_j\|^H \le \|e_0\|^H + 2K \cdot 2K_1$ and $\|e_0\|^H \le \|f_0(g)\|^H + K_1$. Therefore,

$$\begin{aligned}
\|\gamma_{g,s}\|^H &\le (2K+1)(\|e_0\|^H + 2K \cdot 2K_1) \\
&\le (2K+1)(\widetilde{D}_0\|gb_s\|^G + K_1 + 4KK_1) \\
&\le (2K+1)(\widetilde{D}_0 + K_1 + 4KK_1)\|gb_s\|^G,
\end{aligned}$$

where we use that $\|f_0(g)\|^H \le \widetilde{D}_0\|g\|^G = \widetilde{D}_0\|gb_s\|^G$ in the second inequality. We define $\widetilde{D}_1 := (2K+1)(\widetilde{D}_0 + K_1 + 4KK_1)$.

Now assume that f_i has been defined for $i < k \le n$. Let gb be some R-basis element of L_k^G. Then

$$\partial f_{k-1}(\partial gb) = f_{k-2}(\partial \partial gb) = f_{k-2}(0) = 0,$$

so $c := f_{k-1}(\partial gb)$ is a cycle. We have

$$\begin{aligned}
\|c\| &= \|f_{k-1}(\partial gb)\| \le D_{k-1}\|\partial gb\| = D_{k-1}\|\partial b\| \le D_{k-1}C_k, \\
\|c\|^H &= \|f_{k-1}(\partial gb)\|^H \le \widetilde{D}_{k-1}\|\partial gb\|^G \le M_k\widetilde{D}_{k-1}\|gb\|^G,
\end{aligned}$$

where $C_k := \max\{\|\partial b\| \mid b \text{ an } RG\text{-basis element of } L_k^G\}$ and M_k is the constant obtained from Lemma 2.6 applied to $\partial_k \colon L_k^G \to L_{k-1}^G$. Let y be the filling of c

obtained from Lemma 4.5 and define $f_k(gb) := y$. Then

$$\begin{aligned}
\|y\|^H &\leq \|c\|\|c\|^H A_H^k(\|c\|) \\
&\leq \|c\|^H D_{k-1}C_k \cdot A_H^k(D_{k-1}C_k) \\
&\leq \|gb\|^G \underbrace{M_k \widetilde{D}_{k-1} D_{k-1} C_k \cdot A_H^k(D_{k-1}C_k)}_{=:\widetilde{D}_k}
\end{aligned}$$

and

$$\|y\| \leq \|c\| A_H^k(\|c\|) \leq D_{k-1}C_k \cdot A_H^k(D_{k-1}C_k) =: D_k. \qquad \square$$

Lemma 5.2. *Let R be 1-separated. We retain the assumptions of Lemma 5.1 up to degree $n-1$. Let $f\colon G \to H$ and $h\colon H \to G$ be (K,K)-quasi-isometries such that $h \circ f$ is K-close to id_G. Let f_* and h_* be the partial chain maps obtained from Lemma 5.1. Further assume $\mathrm{FV}_{R,G}^n(l) < \infty$ for all l.*

Then for $i \leq n-1$ there exist R-linear maps $s_i\colon L_i^G \to L_{i+1}^G$ and constants E_i, $\widetilde{E}_i$ such that

(1) $\partial s_i + s_{i-1}\partial = \mathrm{id} - h_i \circ f_i$ *for $i \leq n-1$;*
(2) $\|s_i(x)\| \leq E_i\|x\|$ *for all $x \in L_i^G$ and $i \leq n-1$;*
(3) $\|s_i(x)\|^G \leq \widetilde{E}_i\|x\|^G$ *for all $x \in L_i^G$ and $i \leq n-1$.*

Proof. For $g \in G$ let γ_g be a path from g to $h(f(g))$ in $\mathrm{Cay}(G,S)$. Since $h \circ f$ is K-close to id_G, we may choose γ_g such that it has length at most K. We define s_0 to be the R-linear extension such that $s_0(g) = \gamma_g$. Then $\|s_0(g)\| \leq K\|g\|$ and we set $E_0 := K$. Similar to the beginning of the proof of Lemma 5.1 we compute that $\|\gamma_g\|^G \leq \widetilde{E}_1\|g\|^G$, where $\widetilde{E}_1 := K(1 + (K-1) \cdot 2K_1)$.

Assume that we have constructed $s_{i-1}\colon L_{i-1}^G \to L_i^G$ for $i \leq n-1$ and let $c = gb$ be an R-basis element of L_i^G. Note that

$$\partial s_{i-1}(\partial c) = \partial c - \partial h_i(f_i(c)) = \partial\big(c - h_i(f_i(c))\big),$$

hence $c' := c - h_i(f_i(c)) - s_{i-1}(\partial c)$ is a cycle. As in the proof of Lemma 5.1 we set $C_i = \max\{\|\partial b\| \mid b \text{ an } RG\text{-basis element of } L_i^G\}$ to conclude that $\|c'\| \leq |1_R| + D_i'D_i\,|1_R| + E_{i-1}C_i =: F_i$. Define $s_i(c)$ to be the filling y of c' obtained from Lemma 4.5. Then

$$\begin{aligned}
\|y\| &\leq \|c'\| A_G^{i+1}(\|c'\|) \leq F_i A_G^{i+1}(F_i) =: E_i, \\
\|y\|^G &\leq \|c'\|\|c'\|^G A_G^{i+1}(\|c'\|) \leq \underbrace{E_i \cdot (1 + \widetilde{D}_i\widetilde{D}_i' + \widetilde{E}_{i-1}M_i)}_{=:\widetilde{E}_i} \|c\|^G,
\end{aligned}$$

where M_i is the constant obtained from Lemma 2.6 applied to ∂_i. $\qquad\square$

Remark 5.3. The assumption that R is 1-separated is only needed for the bound on the weighted norm in Lemma 4.5. If one is only interested in the non-weighted filling functions one can always construct the respective maps from Lemmas 5.1 and 5.2 only satisfying bounds in the non-weighted norm. Without the need for a bound on the weighted norm Lemma 4.5 can be replaced by the observation that every cycle c admits a filling y such that $\|y\| \leq \mathrm{FV}_{R,\Gamma}^i(\|c\|) + 1$. For details see the proofs of [BKV25, Section 2]. If one works with free resolutions as in Remark 2.8 the proofs can be followed almost verbatim.

Theorem 5.4. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Assume $\mathrm{FV}_{R,G}^i(l), \mathrm{FV}_{R,H}^i(l) < \infty$ for all $2 \le i \le n$ and all l. Then $\mathrm{FV}_{R,G}^n \approx \mathrm{FV}_{R,H}^n$.*

Proof. Let $f \colon G \to H$ be a quasi-isometry with quasi-inverse $h \colon H \to G$, f_*, h_* be the partial chain maps obtained from Lemma 5.1 and D_i, D_i' the associated constants.

Let $s_i \colon L_i^G \to L_{i+1}^G$ and E_i be the maps and constants obtained from Lemma 5.2. If $z \in L_{n-1}^G$ is an $(n-1)$-cycle, so is $f_{n-1}(z)$ and $\|f_{n-1}(z)\| \le D_{n-1}\|z\|$. By assumption, there exists a filling $b \in L_n^H$ of $f_{n-1}(z)$ such that $\|b\| \le \mathrm{FV}_{R,H}^n(D_{n-1}\|z\|) + 1$. Then $h_n(b) + s_{n-1}(z)$ is a filling of z, as

$$\partial h_n(b) + \partial s_{n-1}(z) = h_{n-1}(\partial b) + \partial s_{n-1}(z)$$
$$= h_{n-1}(f_{n-1}(z)) + (z - h_{n-1}f_{n-1}(z) - s_{n-2}(\partial z)) = z.$$

Moreover,

$$\|h_n(b) + s_{n-1}(z)\| \le D_n'\|b\| + E_{n-1}\|z\|$$
$$\le D_n'\left(\mathrm{FV}_{R,H}^n(D_{n-1}\|z\|) + 1\right) + E_{n-1}\|z\|,$$

so $\mathrm{FV}_{R,G}^n \preccurlyeq \mathrm{FV}_{R,H}^n$. The argument is symmetric, completing the proof. □

As a direct consequence of Theorems 5.4 and 4.4 we obtain that $\mathrm{DFV}_{R,G}^n$ is a quasi-isometry invariant for groups of type $\mathrm{FP}_n(R)$.

Theorem 5.5. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{DFV}_{R,G}^n \approx \mathrm{DFV}_{R,H}^n$.*

Remark 5.6. It was proven by Alonso [Alo94] that $\mathrm{FP}_n(R)$ is a quasi-isometry invariant. We could therefore weaken the hypothesis of Theorems 5.4 and 5.5 to just requiring G to be of type $\mathrm{FP}_n(R)$. This is not possible for groups of type $\mathrm{FH}_n(R)$, where it is still an open problem, whether $\mathrm{FH}_n(R)$ is invariant under quasi-isometry. Note that for $n = 2$ Theorem 4.4 is a consequence of [BKV25, 2.21], as $\mathrm{FP}_n(R)$ and $\mathrm{FH}_n(R)$ coincide for $n \le 2$.

In [FMP18], Joshua W. Fleming and Eduardo Martínez-Pedroza proved that for $R = \mathbb{Z}$ equipped with the absolute value $\mathrm{FV}_{\mathbb{Z},G}^n$ takes finite values (Corollary 4.2 gives another proof of this fact). Therefore, the integral homological Dehn functions of G are quasi-isometry invariants for groups of type $\mathrm{FP}_n(\mathbb{Z})$. This has probably already been known to experts in the field, but while there may be a reference for this result, the author is not aware of any. We therefore record it here for the reader's convenience.

Theorem 5.7 ([Fle98, You11, FMP18]). *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(\mathbb{Z})$. Then $\mathrm{FV}_{\mathbb{Z},G}^n \approx \mathrm{FV}_{\mathbb{Z},H}^n$.*

Using Proposition 4.6 we obtain analogous results for the weighted versions of these filling functions.

Theorem 5.8. *Let R be ε-separated. Assume that $\mathrm{wFV}_{R,G}^i(l), \mathrm{wFV}_{R,H}^i(l) < \infty$ for $i \le n$ and all l. If G and H are quasi-isometric, then $\mathrm{wFV}_{R,G}^n \approx \mathrm{wFV}_{R,H}^n$.*

Proof. Due to Lemmas 2.5 and 2.14 we may assume that R is 1-separated. Now the proof of Theorem 5.4 applies verbatim with D_i, D_i', E_i replaced by $\widetilde{D}_i$, $\widetilde{D}_i'$, $\widetilde{E}_i$ and $\|\cdot\|$ replaced by $\|\cdot\|^G$. □

Theorem 5.9. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{wDFV}_{R,G}^n \approx \mathrm{wDFV}_{R,H}^n$.*

Theorem 5.10. *Let G and H be quasi-isometric groups of type $\mathrm{FP}_n(\mathbb{Z})$. Then $\mathrm{wFV}_{\mathbb{Z},G}^n \approx \mathrm{wFV}_{\mathbb{Z},H}^n$.*

Even though there is a close relationship between weighted and unweighted filling functions due to Proposition 4.6, it is unclear whether knowledge of one of the functions allows us to recover the other.

Question 5.11. Are there groups G, H of type FP_n such that $\mathrm{FV}_{\mathbb{Z},G}^n \approx \mathrm{FV}_{\mathbb{Z},H}^n$ but $\mathrm{wFV}_{\mathbb{Z},G}^n \not\approx \mathrm{wFV}_{\mathbb{Z},H}^n$ or vice versa?

6. Cohomology

This section provides an algebraic proof of the quasi-isometry invariance of group cohomology with coefficients in the group ring, originally established by Gersten [Ger93] for finitely presented groups and more generally by Li [Li17] for groups of type $\mathrm{FP}(R)$. The argument presented here follows Gersten's proof in spirit without relying on the groups being finitely presented. As a consequence, we deduce quasi-isometry invariance of the cohomological dimension $\mathrm{cd}_R(G)$ and of being a (Poincaré) duality group for groups of type $\mathrm{FP}(R)$ – results originally due to Sauer [Sau06], respectively Li [Li17]. The proofs use the techniques developed in Section 3 and Section 5.

In the following let G and H be groups of type $\mathrm{FP}(R)$ for a ring R. Furthermore, let L_*^G and L_*^H be free resolutions of R as an RG- respectively RH-module such that all L_i^G and L_i^H are finitely generated. Let B_i be choices of RG-bases of L_i^G.

For this section we consider R to be equipped with the discrete norm. In this case Theorem 4.4 asserts that $\mathrm{DFV}_{R,G}^i$ and $\mathrm{DFV}_{R,H}^i$ are finite functions for all $i \geq 2$.

Lemma 6.1. *Let $f \colon G \to H$ be a map and $\varphi \colon RG^n \to RH^m$ be an R-linear map such that there exists a constant $C > 0$ with $\mathrm{supp}_H(\varphi(gb_i)) \subseteq B_H(f(g), C)$ for all $g \in G$ and $1 \leq i \leq n$. Then $\mathrm{supp}_H(\varphi(c)) \subseteq B_H(f(\mathrm{supp}_G(c)), C)$ for any $c \in RG^n$.*

Proof. Apply the assumption to each basis element in $\mathrm{supp}_R(c)$ to obtain

$$\mathrm{supp}_H(\varphi(c)) \subseteq \bigcup_{gb \in \mathrm{supp}_R(c)} \mathrm{supp}_H(\varphi(gb)) \subseteq \bigcup_{gb \in \mathrm{supp}_R(c)} B_H(f(g), C) = B_H(f(\mathrm{supp}_G(c)), C). \quad \square$$

Definition 6.2. Let $\varphi \colon L \to L'$ be an R-linear map between free R-modules with bases B and B' respectively. We say that φ is *algebraically proper* if for each $y \in B'$ the set

$$I_\varphi(y) := \{x \in B \mid y \in \mathrm{supp}_R(\varphi(x))\}$$

is finite.

Lemma 6.3. *We retain the assumptions of Lemma 6.1. If f is coarsely injective then φ is algebraically proper.*

Proof. Let B be an RG-basis for RG^n and B' an RH-basis for RH^m, such that GB and HB' are the corresponding R-bases, respectively. Consider $y = hb' \in HB'$ and $x = gb \in I_\varphi(y)$. Then $h \in B_H(f(g), C)$, hence $f(g) \in B_H(h, C)$. Therefore, $I_\varphi(y) \subseteq f^{-1}(B_H(h, C)) \cdot B$, which is finite as f is coarsely injective and B is finite. $\quad \square$

Proposition 6.4. *Let $f: G \to H$ be a (K, K)-quasi-isometry and $f_i: L_i^G \to L_i^H$ be the R-linear chain map constructed in Lemma 5.1.*

Then there are constants $C_i \geq 0$ such that $\mathrm{supp}_H(f_i(x)) \subseteq B_H(f(\mathrm{supp}_G(x)), C_i)$ for all $x \in L_i^G$ and all $i \geq 0$.

Proof. In view of Lemma 6.1 it suffices to consider the case $c = gb \in GB_i$. For $i = 0$ we have $\mathrm{supp}_H(f_0(g)) = \{f(g)\}$, so we can take $C_0 = 0$. Assume that the statement holds for $i - 1 \geq 0$. Then

$$
\text{(6.1)} \qquad
\begin{aligned}
\mathrm{supp}_H(f_{i-1}(\partial gb)) &\subseteq B_H(f(\mathrm{supp}_G(g\partial b)), C_{i-1}) \\
&= B_H(f(g\,\mathrm{supp}_G(\partial b)), C_{i-1}).
\end{aligned}
$$

for any $gb \in GB_i$. Because f is a (K, K)-quasi-isometry, we have

$$
d_H(f(g), f(gg')) \leq K d_G(g, gg') + K = K\ell_G(g') + K
$$

for every $g' \in \mathrm{supp}_G(\partial b)$. Set $C_b := \max\{K\ell_G(g') + K \mid g' \in \mathrm{supp}_G(\partial b)\}$. Then

$$
f(g\,\mathrm{supp}_G(\partial b)) \subseteq B_H(f(g), C_b)
$$

and thus, using Eq. (6.1), every $gb \in GB_i$ satisfies

$$
\text{(6.2)} \qquad \mathrm{supp}_H(f_{i-1}(\partial gb)) \subseteq B_H(f(g), C_{i-1} + \widetilde{C_i}),
$$

where $\widetilde{C_i} := \max\{C_b \mid b \in B_i\}$.

Lemma 3.8 yields a constant C_i' such that every $u \in \mathrm{supp}_R(f_i(gb))$ satisfies $d_H(u, v_u) \leq C_i' \|f_i(gb)\|_{\mathrm{d}}$ for some $v_u \in \mathrm{supp}_R(f_{i-1}(\partial gb))$. Here we use that $f_i(gb)$ is reduced. Writing $u = h_u b_u$, and $v_u = h_v b_v$ for $h_u, h_v \in H$, $b_u \in B_i$, and $h_v \in B_{i-1}$ we see that $h_v \in \mathrm{supp}_H(f_i(gb))$ and

$$
\text{(6.3)} \qquad d_H(h_u, h_v) \leq C_i' D_i \|gb\|_{\mathrm{d}} = C_i' D_i,
$$

where D_i is the constant from Lemma 5.1. Together, Eqs. (6.2) and (6.3) yield

$$
\mathrm{supp}_H(f_i(gb)) \subseteq B_H(f(g), C_{i-1} + \widetilde{C_i} + C_i' D_i). \qquad \square
$$

Proposition 6.5. *Let $f: G \to H$ be a (K, K)-quasi-isometry with quasi-inverse $h: H \to G$ such that $h \circ f$ is K-close to id_G. Let $f_i: L_i^G \to L_i^H$, $h_i: L_i^H \to L_i^G$ be the R-linear chain maps constructed in Lemma 5.1. Further, let $s_i: L_i^G \to L_{i+1}^G$ be the R-linear homotopy from Lemma 5.2.*

Then there are constants $A_i \geq 0$ such that for all $x \in L_i^G$

$$
\mathrm{supp}_G(s_i(x)) \subseteq B_G(\mathrm{supp}_G(x), A_i).
$$

Proof. As in the proof of Proposition 6.4 we can restrict to the case $c = gb \in GB_i$. For $i = 0$ an element $g \in G = GB_0$ gets mapped to a geodesic path $s_0(g)$ from g to $h(f(g))$ of length at most K. Corollary 3.6 then tells us that $\mathrm{supp}_G(s_0(g))$ is contained in $B_G(g, 2K_1 K)$.

Now assume that the statement holds for $i - 1 \geq 0$. There exist $C_i > 0$ and $C_i' > 0$ such that

$$
\text{(6.4)} \qquad
\begin{aligned}
\mathrm{supp}_G(h_i(f_i(c))) &\subseteq B_G\Big(h(\mathrm{supp}_H(f_i(c))), C_i'\Big) \\
&\subseteq B_G\Big(h(B_H(f(\mathrm{supp}_G(c)), C_i)), C_i'\Big).
\end{aligned}
$$

for any $c \in L_i^G$. Let $g \in \mathrm{supp}_G(c)$ and $g' \in B_H(f(g), C_i)$. Then

$$d_G(h(g'), g) \leq d_G(h(g'), h(f(g))) + d_G(h(f(g)), g)$$
$$\leq (K \cdot d_H(g', f(g)) + K) + K$$
$$\leq KC_i + 2K,$$

hence $h(B_H(f(g), C_i)) \subseteq B_G(g, KC_i + 2K)$. Now Eq. (6.4) yields

$$\mathrm{supp}_G(h_i(f_i(c))) \subseteq B_G\Big(h(B_H(f(\mathrm{supp}_G(c)), C_i)), C_i'\Big)$$
$$\subseteq \bigcup_{g \in \mathrm{supp}_G(c)} B_G\Big(B_G(g, KC_i + 2K), C_i'\Big)$$
$$= B_G(\mathrm{supp}_G(c), KC_i + 2K + C_i')$$

for every $c \in L_i^G$. For $c = gb \in GB_i$ the cycle $c' = c - h_i(f_i(c)) - s_{i-1}(\partial c)$ satisfies

$$\mathrm{supp}_G(c') \subseteq \{g\} \cup B_G(g, KC_i + 2K + C_i') \cup B_G(g\,\mathrm{supp}(\partial b), A_{i-1})$$
$$\subseteq B_G(g, \max(KC_i + 2K + C_i', \widetilde{A}_i + A_{i-1})),$$

where $\widetilde{A}_i := \max\{\ell_G(g') \mid g' \in \bigcup_{b \in B_i} \mathrm{supp}_G(\partial b_i)\}$. By definition, $s_i(c)$ is a reduced minimal filling of c' and one proceeds as in the last step of the proof of Proposition 6.4 to obtain a constant A_i such that $\mathrm{supp}_G(s_i(c)) \subseteq B_G(g, A_i)$. □

Propositions 6.4 and 6.5 together with Lemma 6.1 imply that the maps constructed in Lemmas 5.1 and 5.2 are algebraically proper, hence we obtain the following corollary.

Corollary 6.6. *Let $f \colon G \to H$ be a quasi-isometry. There exists an R-linear chain homotopy equivalence $f_* \colon L_*^G \to L_*^H$ with chain homotopy inverse $h_* \colon L_*^H \to L_*^G$ and homotopies $s_* \colon h_* \circ f_* \simeq \mathrm{id}$, $t_* \colon f_* \circ h_* \simeq \mathrm{id}$ such that f_i, h_i, s_i and t_i are algebraically proper for all i.*

For a group Γ and a left $R\Gamma$-module M consider the right $R\Gamma$-module $\hom_R^c(M, R)$ consisting of those R-linear maps $f \colon M \to R$ such that, for every $m \in M$, we have $f(\gamma m) = 0$ for all but finitely many $\gamma \in \Gamma$. By [Bro12, VIII Lemma 7.4], there is a natural isomorphism

(6.5) $$\hom_{R\Gamma}(M, R\Gamma) \cong \hom_R^c(M, R).$$

If M is a free $R\Gamma$-module with $R\Gamma$-basis B, the module $\hom_R^c(M, R)$ consists of those maps f for which $f(\gamma b) \neq 0$ for only finitely many $\gamma b \in \Gamma B$.

Evidently, an algebraically proper map $\varphi \colon RG^n \to RH^m$ induces a well-defined map $\hom_R^c(RH^m, R) \to \hom_R^c(RG^n, R)$ through precomposition.

Theorem 6.7 ([Li17, Corollary 4.41])**.** *Let G and H be groups of type $\mathrm{FP}(R)$. If G and H are quasi-isometric then $H^i(G; RG) \cong H^i(H; RH)$ as R-modules for all $i \geq 0$ and any ring R.*

Proof. The chain homotopy equivalence from Corollary 6.6 induces a well-defined R-linear chain homotopy equivalence $\hom_R^c(L_*^H, R) \to \hom_R^c(L_*^G, R)$. Thus,

$$H^i(G; RG) \cong H^i(\hom_{RG}(L_*^G, RG)) \cong H^i(\hom_R^c(L_*^G, R))$$
$$\cong H^i(\hom_R^c(L_*^H, R)) \cong H^i(\hom_{RH}(L_*^H, RH)) \cong H^i(H; RH),$$

where the second and third isomorphisms use Eq. (6.5). □

Corollary 6.8 ([Sau06, Theorem 1.2]). *The cohomological dimension* $\mathrm{cd}_R(G)$ *is a quasi-isometry invariant of groups of type* $\mathrm{FP}(R)$.

Another consequence of Theorem 6.7 is the quasi-isometry invariance of being a (Poincaré) duality group. The proof is the same as the one of Corollary 3 in [Ger93].

Corollary 6.9 ([Ger93, Corollary 3; Li17, Corollary 4.41]). *Let G and H be quasi-isometric groups of type* $\mathrm{FP}(R)$. *If G is a duality group, respectively a Poincaré duality group over R, then so is H.*

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