QUASI-ISOMETRY INVARIANCE OF HOMOLOGICAL DEHN FUNCTIONS

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Abstract. We prove that the nth homological Dehn function with coefficients in an arbitrary normed ring R is a quasi-isometry invariant of groups of type $\mathrm{FP}_n(R)$ for every $n \geq 2$. This confirms a conjecture of Bader–Kropholler–Vankov. The same method proves quasi-isometry invariance of a weighted version of homological Dehn functions.

1. Introduction

For a finitely presented group Γ, the classical *Dehn function* $\delta_\Gamma(l)$ quantifies the difficulty of solving the word problem by measuring the maximal filling area of a null-homotopic word of length at most l. From a computational viewpoint, δ_Γ reflects the complexity of a brute-force approach to the word problem in Γ. This extends naturally to higher dimensions: replacing loops by n-spheres yields the *homotopical* higher Dehn functions δ^n_Γ. It is well known that these functions are quasi-isometry invariants for groups of type F_{n+1}, that is, groups admitting a classifying space with finite $(n+1)$-skeleton [AWP99].

Instead of filling spheres, one can also study the maximal filling volume of integral n-cycles, giving the *homological* higher filling functions $\mathrm{FV}^{n+1}_{\mathbb{Z},\Gamma}$. Homological filling functions have the advantage that they readily generalise to arbitrary normed coefficient rings. This yields a large family of functions $\mathrm{FV}^n_{R,\Gamma}$ depending both on the choice of coefficients and on the chosen norm [LM22, BKV25].

The most common coefficient choices for studying finiteness properties and filling functions – besides the integers with the Euclidean norm – are finite fields equipped with the discrete norm, where the size of a chain is the cardinality of its support [Kro21, LM22, KK21]. More generally, any ring can be given the discrete norm, and we denote the resulting *discrete* filling function by $\mathrm{DFV}^n_{R,\Gamma}$.

A natural question is whether the homological filling functions $\mathrm{FV}^n_{R,\Gamma}$ are also quasi-isometry invariants of the group. In the case of integral filling functions $\mathrm{FV}^n_{\mathbb{Z},\Gamma}$ this is known to be true for groups of type F_n due to Fletcher [Fle98] and Young [You11]. Recent work of Bader, Kropholler, and Vankov [BKV25] shows that $\mathrm{FV}^n_{R,\Gamma}$ is a quasi-isometry invariant for groups of type $\mathrm{FH}_n(R)$, provided the functions take only finite values for both groups. Moreover, they proved that if a group Γ is of type $\mathrm{FH}_n(R)$, then $\mathrm{DFV}^n_{R,\Gamma}$ is finite. They conjectured that quasi-isometry invariance should extend to all groups of type $\mathrm{FP}_n(R)$ and arbitrary normed coefficient rings.

Date: 5th November 2026.
2020 *Mathematics Subject Classification.* 20F69; 20F65; 05C25.
Key words and phrases. Dehn functions, Higher filling functions, Quasi-isometry invariance.

Conjecture (Bader–Kropholler–Vankov [BKV25, Conjecture 2.22])**.** Let $n \geq 2$ and let G and H be quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{FV}^n_{R,G} \approx \mathrm{FV}^n_{R,H}$.

In this work we provide an affirmative answer to this conjecture.

Theorem A. *Suppose that $n \geq 2$ and that G and H are quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{FV}^n_{R,G} \approx \mathrm{FV}^n_{R,H}$.*

Additionally, we give a condition on the norm on R that ensures that $\mathrm{FV}^n_{R,G}$ takes finite values for all groups of type $\mathrm{FP}_n(R)$. A norm $|\cdot|$ on a ring is called ε-separated if $|r| \geq \varepsilon$ for every $r \in R \setminus \{0\}$. In particular, this applies to the discrete norm.

Theorem B. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Assume that the norm on R is ε-separated for some $\varepsilon > 0$. Then $\mathrm{FV}^n_{R,\Gamma}$ takes finite values.*

The separation hypothesis cannot be omitted in general. For every normed ring whose norm is not ε-separated, we exhibit uniformly bounded cycles over $\mathbb{Z}^2$ whose filling volumes tend to infinity; in particular, $\mathrm{FV}^2_{R,\mathbb{Z}^2}$ is not finite (see Subsection 4.4). Related top-dimensional examples in every dimension $n \geq 2$ were recently constructed by Martínez-Pedroza and Vizcaíno Torres [MPT26]. For certain normed rings containing $\mathbb{Q}$, their results show that finiteness of $\mathrm{FV}^n_{R,G}$ already forces linearity.

The construction used for the main theorem also applies to a *weighted* variation of homological filling functions. Any free $R\Gamma$-module $L = \bigoplus_{b\in B} R\Gamma b$ carries an ℓ^1-norm obtained by viewing it as a free R-module with basis ΓB and setting $\left\|\sum a_{\gamma b}\gamma b\right\| = \sum \left|a_{\gamma b}\right|$. In the weighted variant, each basis element γb is assigned the weight $1 + \ell_\Gamma(\gamma)$, where ℓ_Γ denotes the word length in Γ. The resulting norm $\|\cdot\|^\Gamma$ and the corresponding weighted filling functions $\mathrm{wFV}^n_{R,\Gamma}$ appear naturally in several situations. They were first studied by Ogle [Ogl05] and Ji–Ramsey [JR09] in connection with the Rapid Decay property. They were also utilised by Bader–Sauer in their work on higher property (T) [BS23] to deduce isomorphisms in polynomial cohomology under the assumption that $\mathrm{wFV}^n_{\mathbb{Z},\Gamma}$ is polynomially bounded. Furthermore, weighted filling functions are used in joint work with Roman Sauer to give a characterization for polynomiality of filling functions using a homological algebra framework [SW26].

Our techniques for non-weighted filling functions extend to this setting, yielding weighted versions of Theorems 4.3 and 5.10.

Theorem C. *Suppose that $n \geq 2$ and that G and H are quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{wFV}^n_{R,G} \approx \mathrm{wFV}^n_{R,H}$.*

Theorem D. *Let $n \geq 2$ and let Γ be a group of type $\mathrm{FP}_n(R)$. Assume that the norm on R is ε-separated for some $\varepsilon > 0$. Then $\mathrm{wFV}^n_{R,\Gamma}$ takes finite values.*

Proof strategy. The central idea is to equip the R-basis of a free chain complex with a graph structure that records how basis elements interact through their boundaries. This provides an algebraic analogue of the geometry of a cell complex, allowing us, for example, to reason about chains with connected support. In this way, geometric arguments can be transferred to the algebraic setting.

Structure. The paper is organised as follows. In Section 2 we recall the definition of filling functions over normed rings and establish basic properties needed throughout.

Section 3 develops the algebraic counterpart of geometric support arguments on Γ-complexes. In Section 4 we establish the finiteness results for separated norms, compare weighted and unweighted filling functions, and give the non-finite example. In Section 5 we construct admissible chain maps and homotopies and prove the quasi-isometry invariance of ordinary and weighted filling functions. Section 6 gives an alternative proof of quasi-isometry invariance of cohomology with group-ring coefficients using the constructions of Section 5.

Acknowledgements. I would like to thank my advisors Claudio Llosa Isenrich and Roman Sauer for their valuable comments and suggestions during the preparation of this work and Kevin Li for suggesting the content of Section 6 as a possible application. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Research Training Group DFG 281869850 (RTG 2229).

2. Homological filling functions over normed rings

Following [BKV25] and [KK21], we give a short introduction to homological filling functions over arbitrary normed rings.

Throughout, let Γ be a group and R a ring with unit $1 \neq 0$. All modules considered are left modules unless indicated otherwise.

2.1. Normed rings.

Definition 2.1. A *norm* on a ring R is a function $|\cdot| : R \to \mathbb{R}_{\geq 0}$ such that

(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
(ii) $|x + y| \leq |x| + |y|$;
(iii) $|xy| \leq |x|\,|y|$.

A ring together with a choice of norm is called a *normed ring*. Two norms $|\cdot|_1, |\cdot|_2$ are said to be *equivalent* if there exists a constant $C > 0$ such that

$$\frac{1}{C} \cdot |x|_2 \leq |x|_1 \leq C \cdot |x|_2$$

for all $x \in R$.

Example 2.2.

(1) If R is a subring of $\mathbb{C}$, the restriction of the absolute value turns R into a normed ring.
(2) Any ring can be made into a normed ring by equipping it with the *discrete norm*, given by
$$|x|_{\mathrm{d}} = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$
(3) Let Γ be a finitely generated group with word metric d_Γ and associated word length $\ell_\Gamma(\gamma) = d_\Gamma(\gamma, 1_\Gamma)$. Let R be a normed ring. The associated *weighted norm* on the group ring $R\Gamma$ is defined as
$$\left\| \sum_{\gamma \in \Gamma} a_\gamma \gamma \right\|^\Gamma := \sum_{\gamma \in \Gamma} |a_\gamma| \, (1 + \ell_\Gamma(\gamma)).$$
It is a straightforward computation that this is a norm on $R\Gamma$ and the equivalence class of $\|\cdot\|^\Gamma$ does not depend on the chosen word metric.

For any normed ring R, there is an induced ℓ^1-*norm* on any free R-module with basis B given by

$$\left\|\sum_{b\in B}\alpha_b b\right\| = \sum_{b\in B}|\alpha_b|\,.$$

We can consider the group ring $R\Gamma$ as a free R-module with basis Γ and equip it with the ℓ^1-norm.

Let Γ be finitely generated. On free $R\Gamma$-modules L we consider two separate norms. The ℓ^1-norm $\|\cdot\|$ induced by viewing L as a free R-module and the weighted norm $\|\cdot\|^\Gamma$ coming from the weighted norm on $R\Gamma$.

If R is equipped with the discrete norm, we write $\|\cdot\|_{\mathrm{d}}$ and $\|\cdot\|_{\mathrm{d}}^\Gamma$ for the corresponding induced norms.

The following two properties of a ring are sometimes convenient to assume.

Definition 2.3. A norm on a ring is called *symmetric* if $|-x| = |x|$ for all $x \in R$. A norm is called ε-*separated* for $\varepsilon > 0$ if $|x| \geq \varepsilon$ for all $x \in R \setminus \{0\}$.

Examples of symmetric, ε-separated norms include the absolute value norm on $\mathbb{Z}$ and the discrete norm on any ring. For both we can choose $\varepsilon = 1$. In Subsection 4.4, we give an example showing that the assumption of an ε-separated norm is essential for $\mathrm{FV}^n_{R,\Gamma}$ to take finite values for all groups.

Lemma 2.4. *Every norm on a ring R is equivalent to a symmetric norm.*

Proof. One easily checks that $|r|' \coloneqq |r| + |-r|$ defines a norm on R. By definition, this norm is symmetric and $|r| \leq |r|' \leq |r| + |-1|\,|r| = (1 + |-1|)\,|r|$. □

From now on, all norms on rings are assumed to be symmetric.

Lemma 2.5. *Let R be an ε-separated normed ring. Then up to equivalence of norms we may assume that R is* 1*-separated.*

Proof. Scaling a norm by a factor $C \geq 1$ defines a norm which is equivalent to the original one. Therefore, if $\varepsilon < 1$ in Definition 2.3, we can scale the norm by $\frac{1}{\varepsilon}$. □

The following lemma is proved in [BKV25, Lemma 2.3] for the non-weighted ℓ^1-norm on free modules. In the weighted case the proof is identical.

Lemma 2.6. *Let Γ be a finitely generated group and $f\colon M \to N$ be an $R\Gamma$-linear map between finitely generated free $R\Gamma$-modules. Then there exists a constant $C > 0$ such that $\|f(x)\|^\Gamma \leq C\|x\|^\Gamma$ for all $x \in M$.*

2.2. **Filling functions.**

Definition 2.7. Let $n \geq 0$. A group Γ is of type $\mathrm{FP}_n(R)$ if there exists a projective $R\Gamma$-resolution $P_* \to R$ of the trivial $R\Gamma$-module R, where P_i is finitely generated for $i \leq n$. By [Bro12, Proposition VIII.4.3], one may equivalently require the resolution to consist of free modules. If the resolution P_* can be chosen to be finite, that is, $P_i = 0$ for $i > n$ for some n, we say that Γ is of type $\mathrm{FP}(R)$.

Remark 2.8. Let Γ be of type $\mathrm{FP}_n(R)$ with $n \geq 1$. Then there exists a free $R\Gamma$-resolution L_* of the trivial $R\Gamma$-module R such that L_i is finitely generated for $i \leq n$. We may assume that the modules L_1 and L_0 are the cellular chain modules of a Cayley graph of Γ. That is, if S is a finite generating set for Γ, then we may choose L_* to look like the following chain complex

$$\cdots \to L_2 \longrightarrow R\Gamma^{|S|} \xrightarrow{e_i \mapsto (s_i - 1)} R\Gamma \xrightarrow{\varepsilon} R.$$

This follows from the fact that type $\mathrm{FP}_1(R)$ is equivalent to finite generation (see [Bro12, Exercise VIII.4.1]).

Definition 2.9 ([KK21, Definition 2.1])**.** Let $n \geq 0$. We say that a projective $R\Gamma$-resolution $P_* \to R$ is *n-admissible* if P_n and P_{n+1} are finitely generated free $R\Gamma$-modules equipped with a choice of free bases.

The following definition is based on [BKV25, Definitions 2.9, 2.13].

Definition 2.10. Let R be a normed ring and Γ be a group of type $\mathrm{FP}_{n+1}(R)$. Let $P_* \to R$ be an n-admissible resolution. The *filling volume* of a cycle $c \in P_n$ is

$$\mathrm{FV}_R(c) := \inf_{\substack{b \in P_{n+1} \\ \partial b = c}} \|b\|.$$

The *weighted filling volume* of c is defined as

$$\mathrm{wFV}_R(c) := \inf_{\substack{b \in P_{n+1} \\ \partial b = c}} \|b\|^\Gamma.$$

The *nth homological filling function of Γ with coefficients in R* is the function $f_n^{R,\Gamma}(l) \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined through

$$f_n^{R,\Gamma}(l) := \sup_{\substack{c \in P_n \\ \partial c = 0, \|c\| \leq l}} \mathrm{FV}_R(c).$$

The weighted version is defined similarly by

$$\mathrm{w}f_n^{R,\Gamma}(l) := \sup_{\substack{c \in P_n \\ \partial c = 0, \|c\|^\Gamma \leq l}} \mathrm{wFV}_R(c).$$

Definition 2.11. Given functions $f, g \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ we write $f \preccurlyeq g$ if there are constants $C, D > 0$ such that $f(v) \leq Cg(Cv + D) + Cv + D$ for every $v \geq 0$. We write $f \approx g$ if both $f \preccurlyeq g$ and $g \preccurlyeq f$ hold.

The following lemma from [BKV25] tells us that up to $\approx$-equivalence $f_n^{R,\Gamma}$ and $\mathrm{w}f_n^{R,\Gamma}$ are well-defined notions. There it is proven for non-weighted filling functions, however the proof also applies if one replaces all norms by their weighted versions.

Lemma 2.12 ([BKV25, Lemma 2.12])**.** *Let P_* and P'_* be two n-admissible resolutions of the trivial $R\Gamma$-module R. Let f_n and f'_n denote the corresponding (weighted) filling functions from Definition 2.10, respectively. Then $f_n \approx f'_n$.*

Convention 2.13. For a ring R equipped with the discrete norm, the notation $\mathrm{DFV}^n_{R,\Gamma}$ is standard for $f^{R,\Gamma}_{n-1}$. In this case we also write $\mathrm{wDFV}^n_{R,\Gamma}$ for the weighted version. Note that there is an index shift between the two notations. In the following we use the index convention matching $\mathrm{DFV}^n_{R,\Gamma}$. To avoid confusion we write $\mathrm{FV}^n_{R,\Gamma}$ and $\mathrm{wFV}^n_{R,\Gamma}$ for $f^{R,\Gamma}_{n-1}$ and $\mathrm{w}f^{R,\Gamma}_{n-1}$ respectively.

Lemma 2.14. *Equivalent norms on R induce equivalent (weighted) filling functions.*

Proof. We show this for the non-weighted version. The proof is identical in the weighted case. Let $|\cdot|$, $|\cdot|'$ be equivalent norms and $C > 0$ such that $C^{-1}|x|' \leq |x| \leq C|x|'$ for all $x \in R$. We write R' for R equipped with the norm $|\cdot|'$ and

$\|\cdot\|'$ for the ℓ^1-norm induced by $|\cdot|'$. Clearly $C^{-1}\cdot\|c\|' \leq \|c\| \leq C\cdot\|c\|'$ and $C^{-1}\cdot \mathrm{FV}_{R'}(c) \leq \mathrm{FV}_R(c) \leq C\cdot \mathrm{FV}_{R'}(c)$. Therefore,

$$\begin{aligned}\mathrm{FV}^n_{R,\Gamma}(l) &= \sup\{\mathrm{FV}_R(c) \mid \|c\| \leq l\}\\ &\leq \sup\{C\cdot \mathrm{FV}_{R'}(c) \mid \|c\|' \leq C\cdot l\} = C\cdot \mathrm{FV}^n_{R',\Gamma}(C\cdot l),\end{aligned}$$

hence $\mathrm{FV}^n_{R,\Gamma} \preccurlyeq \mathrm{FV}^n_{R',\Gamma}$ and by symmetry $\mathrm{FV}^n_{R',\Gamma} \approx \mathrm{FV}^n_{R,\Gamma}$. □

3. Cellular arguments in free chain-complexes

The goal of this section is to develop tools for translating geometric arguments in cell complexes to the setting of free chain complexes.

In the following, let Γ be a finitely generated group with a word metric d_Γ and associated word length $\ell_\Gamma(\gamma) = d_\Gamma(\gamma, 1_\Gamma)$. Let $n \in \mathbb{N}\cup\{\infty\}$ and let L_* be a non-negative free $R\Gamma$-chain complex such that L_i is finitely generated for $i \leq n$. For each $i \leq n$ let B_i be a finite $R\Gamma$-basis of L_i. Then $\Gamma B_i = \{\gamma b \mid \gamma \in \Gamma,\, b \in B_i\}$ is an R-basis for L_i.

3.1. **A geometric structure on chain complexes.** Let $\alpha = \sum_{u\in\Gamma B_i} a_u u \in L_i$. The *$R$-support* of α is the set

$$\mathrm{supp}_R(\alpha) \coloneqq \{u \in \Gamma B_i \mid a_u \neq 0\}.$$

Similarly, we define the *Γ-support* of α as

$$\mathrm{supp}_\Gamma(\alpha) \coloneqq \{\gamma \in \Gamma \mid a_{\gamma b} \neq 0 \text{ for some } b \in B_i\}.$$

The functions d_Γ and ℓ_Γ can be extended to

$$d_\Gamma\colon \coprod_{i=0}^n \Gamma B_i \times \coprod_{i=0}^n \Gamma B_i \to \mathbb{N}_0, \quad \ell_\Gamma\colon \coprod_{i=0}^n \Gamma B_i \to \mathbb{N}_0$$

by setting $d_\Gamma(\gamma b, \gamma' b') \coloneqq d_\Gamma(\gamma,\gamma')$ and $\ell_\Gamma(\gamma b) \coloneqq \ell_\Gamma(\gamma)$. Note that d_Γ is only a pseudo-metric on $\coprod_i \Gamma B_i$. For a basis element $x = \gamma b$, write $\omega_\Gamma(x) \coloneqq 1 + \ell_\Gamma(\gamma)$ for its weight.

Remark 3.1. Let $x = \gamma_x b \in \Gamma B_i$ and $y = \gamma_y b' \in \Gamma B_j$. Then

$$\ell_\Gamma(y) = \ell_\Gamma(\gamma_y) \leq \ell_\Gamma(\gamma_x) + d_\Gamma(\gamma_x,\gamma_y) = \ell_\Gamma(x) + d_\Gamma(x,y),$$

hence $\omega_\Gamma(y) \leq \omega_\Gamma(x) + d_\Gamma(x,y)$.

Definition 3.2. Let U be a subset of the R-basis ΓB_i. We define a non-oriented graph $\mathrm{Gr}(U)$ with vertex set U and edges given by $\{u,v\}$ for $u \neq v \in U$ such that $\mathrm{supp}_R(\partial u) \cap \mathrm{supp}_R(\partial v) \neq \varnothing$. For an element $\alpha \in L_k$ we define $\mathrm{Gr}(\alpha)$ as $\mathrm{Gr}(\mathrm{supp}_R(\alpha))$. We say that U, respectively α, is *connected* if $\mathrm{Gr}(U)$, respectively $\mathrm{Gr}(\alpha)$, is connected.

Remark 3.3. For $V \subset U$ the graph $\mathrm{Gr}(V)$ is the induced subgraph of $\mathrm{Gr}(U)$ on the vertices V. Therefore, if U is not connected, there exists a decomposition $U = \bigsqcup_j U_j$ such that each U_j is connected and of maximal size corresponding to a connected component of $\mathrm{Gr}(U)$. In particular, if $\alpha \in L_i$ is not connected, this decomposition of $\mathrm{supp}_R(\alpha)$ induces a decomposition $\alpha = \sum_{j=1}^m \alpha_j$ such that $\|\alpha\| = \sum_{j=1}^m \|\alpha_j\|$. If $i \geq 1$ and α is a cycle, then each α_j is also a cycle, as no cancellation can happen between different connected components.

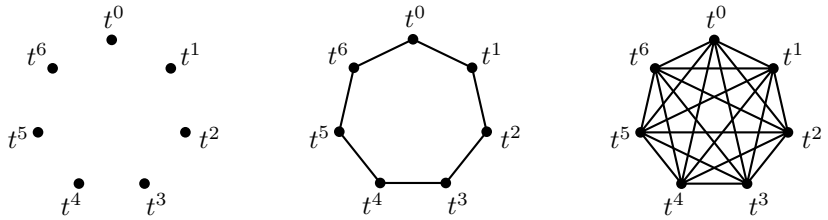


FIGURE 1. The graphs associated to the chain modules of the standard free $R\Gamma$-resolution of $\Gamma = \mathbb{Z}/7\mathbb{Z}$.

Example 3.4. If $\Gamma = \mathbb{Z}/k\mathbb{Z} = \langle t \mid t^k \rangle$, a free $R\Gamma$-resolution of R is given by

$$\cdots \xrightarrow{N\cdot} R\Gamma \xrightarrow{(1-t)\cdot} R\Gamma \xrightarrow{N\cdot} R\Gamma \xrightarrow{(1-t)\cdot} R\Gamma \xrightarrow{\varepsilon} R,$$

where $N = \sum_{i=0}^{k-1} t^i$. For each i a free R-basis of $R\Gamma$ is $\Gamma B_i = \{t^j \mid 0 \leq j \leq k-1\}$. We have $N \cdot t^j = N$ and $(1-t) \cdot t^j = t^j - t^{j+1}$. Therefore, $\mathrm{Gr}(\Gamma B_n)$ is a cycle on k vertices if n is odd and a complete graph if n is even, except for $n = 0$, as $\mathrm{Gr}(\Gamma B_0)$ does not contain any edges (see Figure 1).

Assume that L_* is the cellular chain complex of a free simplicial Γ-complex X. Then ΓB_i corresponds bijectively to the set of i-cells of X, and $\mathrm{Gr}(\Gamma B_i)$ is the graph whose edges connect cells sharing a common face. We endow X with the metric induced by the Euclidean metric on each simplex, which in turn defines a metric on the sets of simplices via the distance between their barycentres. This allows us to bound the distance between neighbouring cells. The next lemma provides an algebraic analogue of this phenomenon.

Lemma 3.5. *Let $0 \leq i \leq n$ and $U \subseteq \Gamma B_i$. There exists $K_i \geq 0$ such that*

1. *if $x \in U$ and $v \in \mathrm{supp}_R(\partial x)$, then $d_\Gamma(x, v) \leq K_i$ and $\omega_\Gamma(x) \leq \omega_\Gamma(v) + K_i$;*
2. *if $x, y \in U$ are connected by an edge in $\mathrm{Gr}(U)$, then $d_\Gamma(x, y) \leq 2K_i$ and $\omega_\Gamma(y) \leq \omega_\Gamma(x) + 2 \cdot K_i$.*

Proof. In both cases we just show the estimate on the distance. The statement for the weight then follows from Remark 3.1. Consider the set $A_i := \bigcup_{b^{(i)} \in B_i} \mathrm{supp}_\Gamma(\partial b^{(i)})$. If A_i is empty then either B_i is empty or $\partial_i = 0$. In either case both statements of the lemma are vacuously true, so we may set $K_i = 0$. From now on, let us assume that A_i is non-empty and define $K_i := \max\{\ell_\Gamma(h) \mid h \in A_i\} < \infty$.

Let $x = \gamma_x b_x^{(i)} \in U$ and $v = \gamma_v b_v^{(i-1)} \in \mathrm{supp}_R(\partial x)$. Since $\partial x = \gamma_x \partial b_x^{(i)}$, we can write $\gamma_v = \gamma_x h$ for some $h \in \mathrm{supp}_\Gamma(\partial b_x^{(i)}) \subseteq A_i$. Hence, $d_\Gamma(x, v) = \ell_\Gamma(h) \leq K_i$.

Now let $y = \gamma_y b_y^{(i)}$ be another element in U such that x and y are connected by an edge in $\mathrm{Gr}(U)$. Then there exist $h_x, h_y \in A_i$ and an $R\Gamma$-basis element $b^{(i-1)} \in B_{i-1}$ such that

$$\gamma_x h_x b^{(i-1)} = \gamma_y h_y b^{(i-1)} \in \mathrm{supp}_R(\partial x) \cap \mathrm{supp}_R(\partial y).$$

Hence, $\gamma_y = \gamma_x h_x h_y^{-1}$ and therefore $d_\Gamma(x, y) \leq \ell_\Gamma(h_x) + \ell_\Gamma(h_y) \leq 2K_i$. □

Corollary 3.6. *Let $0 \leq i \leq n$, $c \in L_i$ be connected, and let $x, y \in \mathrm{supp}_R(c)$.*

1. *$d_\Gamma(x, y) \leq \left|\mathrm{supp}_R(c)\right| \cdot 2K_i$ and $\omega_\Gamma(y) \leq \omega_\Gamma(x) + \left|\mathrm{supp}_R(c)\right| \cdot 2K_i$;*

Assume, in addition, that R is 1-separated. Then

(2) $\omega_\Gamma(y) \leq \omega_\Gamma(x) + 2K_i\|c\|$ *and* $\|c\|^\Gamma \leq (\omega_\Gamma(x) + 2K_i\|c\|)\|c\|$.

Proof. The diameter of $\mathrm{Gr}(c)$ is at most $\left|\mathrm{supp}_R(c)\right|$, hence the first claim follows from Lemma 3.5. If R is 1-separated, $\|c\| \geq \left|\mathrm{supp}_R(c)\right|$. Writing $c = \sum_{\gamma b \in \Gamma B_i} \alpha_{\gamma b}\gamma b$ we obtain

$$\begin{aligned}\|c\|^\Gamma = \sum_{\gamma b} \left|\alpha_{\gamma b}\right| \omega_\Gamma(\gamma b) &\leq \sum_{\gamma b} \left|\alpha_{\gamma b}\right| \left(\omega_\Gamma(x) + \left|\mathrm{supp}_R(c)\right| 2K_i\right) \\ &\leq (\omega_\Gamma(x) + 2K_i\|c\|)\|c\|. \qquad \square\end{aligned}$$

Definition 3.7. Let $\alpha \in L_i$. We say that α is *reduced* if α does not admit a decomposition $\alpha = \alpha_1 + \alpha_2$ such that $\alpha_1 \neq 0$, $\partial\alpha_1 = 0$, and $\mathrm{supp}_R(\alpha_1) \cap \mathrm{supp}_R(\alpha_2) = \varnothing$.

Remark 3.8. Every chain α can be replaced by a reduced chain with the same boundary and no larger norm. Indeed, if $\alpha = \alpha_1 + \alpha_2$ witnesses that α is not reduced, then $\partial\alpha_2 = \partial\alpha$ and the disjointness of the supports gives $\|\alpha\| = \|\alpha_1\| + \|\alpha_2\|$ and $\|\alpha\|^\Gamma = \|\alpha_1\|^\Gamma + \|\alpha_2\|^\Gamma$. Replacing α by α_2 strictly decreases the finite set $\mathrm{supp}_R(\alpha)$, so iterating this procedure terminates.

Lemma 3.9. *There exists a constant $C_i \geq 0$ such that for every reduced chain $\alpha \in L_i$ and every $u \in \mathrm{supp}_R(\alpha)$ there is $v_u \in \mathrm{supp}_R(\partial\alpha)$ with $d_\Gamma(u, v_u) \leq C_i \left|\mathrm{supp}_R(\alpha)\right|$ and thus $\omega_\Gamma(u) \leq \omega_\Gamma(v_u) + C_i\|\alpha\|_{\mathrm{d}}$.*

Proof. Decompose α into its connected components $\alpha = \alpha_1 + \cdots + \alpha_k$. Because α is reduced, $\partial\alpha_p \neq 0$ for every $1 \leq p \leq k$. Hence, we can pick vertices u_p of $\mathrm{Gr}(\alpha_p)$ such that $\mathrm{supp}_R(\partial u_p) \cap \mathrm{supp}_R(\partial\alpha_p) \neq \varnothing$. Let $v_p \in \mathrm{supp}_R(\partial u_p) \cap \mathrm{supp}_R(\partial\alpha_p)$ be such an element in the intersection. By Lemma 3.5 and Corollary 3.6, every $u \in \mathrm{supp}_R(\alpha_p)$ satisfies

$$\begin{aligned} d_\Gamma(u, v_p) \leq d_\Gamma(u, u_p) + d_\Gamma(u_p, v_p) \leq 2K_i \left|\mathrm{supp}_R(\alpha_p)\right| + K_i \leq 3K_i \left|\mathrm{supp}_R(\alpha)\right|, \\ \omega_\Gamma(u) \leq \omega_\Gamma(u_p) + \left|\mathrm{supp}_R(\alpha_p)\right| \cdot 2K_i \leq \omega_\Gamma(v_p) + K_i + \left|\mathrm{supp}_R(\alpha)\right| \cdot 2K_i. \end{aligned}$$

Thus, we may take $C_i \coloneqq 3K_i$. $\square$

Lemma 3.10. $\mathrm{Gr}(\Gamma B_i)$ *is locally finite for all $0 \leq i \leq n$.*

Proof. Because the differentials of L_* are Γ-equivariant, Γ acts on $\mathrm{Gr}(\Gamma B_i)$ by graph automorphisms. It therefore suffices to show that

$$B_{\mathrm{Gr}(\Gamma B_i)}(b, 1) = \{b\} \cup \{\gamma b' \in \Gamma B_i \mid \{\gamma b', b\} \text{ is an edge in } \mathrm{Gr}(\Gamma B_i)\}$$

is finite for every $b \in B_i$. To this end let $\gamma b' \in \Gamma B_i$ be connected to b by an edge in $\mathrm{Gr}(\Gamma B_i)$. Then by Lemma 3.5 we have that

$$\ell_\Gamma(\gamma) = \ell_\Gamma(\gamma b') \leq \ell_\Gamma(b) + 2K_i = 2K_i$$

and hence $\ell_\Gamma(\gamma) \leq 2K_i$. As Γ is finitely generated and B_i is finite, we conclude that $B_{\mathrm{Gr}(\Gamma B_i)}(b, 1)$ is finite. $\square$

For $i = 0$ the above statement is trivially satisfied, as B_{-1} is empty and therefore $\mathrm{Gr}(\Gamma B_0)$ contains no edges at all.

We now study algebraic analogues of subcomplexes of a simplicial Γ-complex. For $l \geq 0$ consider the collection

$$\mathcal{C}_l^i := \{U \subseteq \Gamma B_i \mid |U| \leq l,\ U \text{ connected}\}.$$

An element $U \in \mathcal{C}_l^i$ can be thought of as giving a basis to a connected R-subspace of L_i with R-dimension at most l. The group Γ acts on $\mathcal{C}_l^i$ by left multiplication, that is, $\gamma U := \{\gamma u \mid u \in U\}$.

Proposition 3.11. *Let $l \geq 0$ and $0 \leq i \leq n$. Then $\Gamma\backslash\mathcal{C}_l^i$ is finite.*

Proof. Let $U \in \mathcal{C}_l^i$. If $U = \varnothing$, the orbit ΓU just consists of U. If $U \neq \varnothing$ let $\gamma b \in U$. By considering $U' := \gamma^{-1}U$, we may assume that $\gamma = 1$ and $b \in U$. Because $|U| \leq l$, the diameter of $\mathrm{Gr}(U)$ is at most l. Repeated application of Lemma 3.5 yields that for any $\gamma' b' \in U$ we have

$$\ell_\Gamma(\gamma') = \ell_\Gamma(\gamma' b') \leq \ell_\Gamma(b) + 2K_i l = 2K_i l.$$

Therefore, U is a subset of the finite set $\{\gamma \in \Gamma \mid \ell_\Gamma(\gamma) \leq 2K_i l\}B_i$. □

4. Finiteness of filling functions

In this section we prove finiteness for various filling functions introduced in Subsection 2.2.

4.1. **Integral filling functions.** The following proposition recovers the result that $\mathrm{FV}^n_{\mathbb{Z},\Gamma}$ takes finite values [FMP18, Theorem 1.3].

Proposition 4.1. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Assume that for each $l \geq 0$ the set $R_{\leq l} := \{r \in R \mid |r| \leq l\}$ is finite. Then $\mathrm{FV}^n_{R,\Gamma}$ takes finite values.*

Proof. Fix a free $R\Gamma$-resolution $L_* \to R$ such that L_i is finitely generated for $i \leq n$, and fix finite $R\Gamma$-bases in these degrees. We consider the collection

$$\mathcal{D}_l^{n-1} := \{c \in L_{n-1} \setminus \{0\} \mid \partial c = 0,\ c \text{ connected}, \|c\| \leq l\}$$

of connected non-zero $(n-1)$-cycles with norm at most l. The group Γ acts on $\mathcal{D}_l^{n-1}$ by left multiplication, as $\|\gamma c\| = \|c\| \leq l$ for each $c \in \mathcal{D}_l^{n-1}$ and $\gamma \in \Gamma$. By assumption, $R_{\leq 1}$ is finite, hence R is ε-separated for some $\varepsilon > 0$. Using Lemma 2.5 we may assume that $\varepsilon = 1$, hence $\left|\mathrm{supp}_R(c)\right| \leq \|c\| \leq l$. Let $U_1, \ldots, U_N$ be representatives of $\Gamma\backslash\mathcal{C}_l^{n-1}$. If $c \in \mathcal{D}_l^{n-1}$ is a cycle with $\mathrm{supp}_R(c) = U_j$, it is contained in the set

$$R_{\leq l}U_j := \Big\{\sum_{u \in U_j} \alpha_u u \mid \alpha_u \in R_{\leq l}\Big\},$$

which is finite as $R_{\leq l}$ and U_j are finite. We conclude that $\Gamma\backslash\mathcal{D}_l^{n-1}$ is finite. This implies that there is some $D(l) \geq 0$ such that $\mathrm{FV}_R(c) \leq D(l)$ for every $c \in \mathcal{D}_l^{n-1}$. In the general case a cycle c decomposes as a sum $c = c_1 + \cdots + c_s$ of non-zero connected cycles. Then $\|c\| = \sum_i \|c_i\|$, and 1-separation implies $s \leq \|c\| \leq l$. For each i, choose a filling b_i with $\|b_i\| \leq D(l) + 1$. Then $\sum_i b_i$ is a filling of c, and $\left\|\sum_i b_i\right\| \leq sD(l) + s \leq lD(l) + l$. Therefore, $\mathrm{FV}^n_{R,\Gamma}(l) \leq lD(l) + l < \infty$. □

Corollary 4.2. *Let $n \geq 2$ and let Γ be a group of type $\mathrm{FP}_n(\mathbb{Z})$. $\mathrm{FV}^n_{\mathbb{Z},\Gamma}$ takes finite values.*

4.2. **Separated filling functions.** The finiteness result from the previous subsection extends to every ε-separated norm.

Theorem 4.3. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Assume that the norm on R is ε-separated for some $\varepsilon > 0$. Then $\mathrm{FV}^n_{R,\Gamma}$ takes finite values.*

Proof. By Lemmas 2.5 and 2.14, we may assume that the norm on R is 1-separated. Let $L_* \to R$ be a free $R\Gamma$-resolution such that L_i is finitely generated for $i \leq n$, and fix finite $R\Gamma$-bases B_i in these degrees.

After forgetting the Γ-action, $L_* \to R$ is a free resolution of the free R-module R and is therefore R-linearly contractible. Let $s_i \colon L_i \to L_{i+1}$ be an R-linear chain contraction. Thus, $\partial_n s_{n-1}(c) = c$ for every $(n-1)$-cycle c.

For a 1-separated norm, $\|c\| < 1$ already implies that $c = 0$. It therefore suffices to consider the case where $\|c\| \leq l$ for $l \geq 1$. By Proposition 3.11, there are $U_1, \dots, U_m \in \mathcal{C}^{n-1}_l \setminus \{\varnothing\}$ representing all Γ-orbits in $\mathcal{C}^{n-1}_l \setminus \{\varnothing\}$. Define

$$D(l) \coloneqq \max\{\|s_{n-1}(u)\| \mid u \in U_i,\ 1 \leq i \leq m\} < \infty.$$

Let c be a non-zero connected $(n-1)$-cycle with $\|c\| \leq l$. Since the norm is 1-separated, $\left|\mathrm{supp}_R(c)\right| \leq \|c\| \leq l$. Hence, there are i and $\gamma \in \Gamma$ such that $\mathrm{supp}_R(c) = \gamma U_i$. Set $c' \coloneqq \gamma^{-1}c$ and $b \coloneqq \gamma s_{n-1}(c')$. Then, $\partial_n b = \gamma \partial_n s_{n-1}(c') = \gamma c' = c$, so b is a filling of c. Writing $c' = \sum_{u \in U_i} \alpha_u u$, we obtain

$$\|b\| = \|s_{n-1}(c')\| \leq \sum_{u \in U_i} |\alpha_u| \, \|s_{n-1}(u)\| \leq D(l)\|c'\| = D(l)\|c\|.$$

In the non-connected case, decompose the cycle into its connected components as in the proof of Proposition 4.1. Since $n \geq 2$, every component is a cycle, and the sum of their norms is $\|c\|$. Applying the estimate above to every component gives $\mathrm{FV}^n_{R,\Gamma}(l) \leq D(l)l < \infty$. □

Corollary 4.4. *Let $n \geq 2$ and Γ be a group of type $\mathrm{FP}_n(R)$. Then $\mathrm{DFV}^n_{R,\Gamma}$ takes finite values.*

Proof. The discrete norm is 1-separated, so the claim follows from Theorem 4.3. □

4.3. **Weighted filling functions.** In the following let Γ be a group of type $\mathrm{FP}_n(R)$. We fix a free resolution $L_* \to R$ and finite $R\Gamma$-bases of L_i for $i \leq n$.

Lemma 4.5. *Let $2 \leq i \leq n$. Assume that R is 1-separated and $\mathrm{wFV}^i_{R,\Gamma}(l) < \infty$ for all l. For any $(i-1)$-cycle $c \in L_{i-1}$ there exists a reduced filling y such that*

$$\|y\|^\Gamma \leq \|c\|\|c\|^\Gamma A^i_\Gamma(\|c\|) \quad \text{and} \quad \|y\| \leq \|c\| A^i_\Gamma(\|c\|),$$

where K_{i-1} is the constant from Lemma 3.5 and

$$A^i_\Gamma(x) = \mathrm{wFV}^i_{R,\Gamma}(x + 2K_{i-1}x^2) + 1.$$

Proof. Because R is 1-separated, c admits a decomposition $c = c_1 + \cdots + c_m$ into connected cycles with $m \leq \|c\|$. For $h_j \in \mathrm{supp}_\Gamma(c_j)$ consider the cycles $c'_j \coloneqq h_j^{-1}c_j$. Choose fillings y'_j of c'_j such that

$$\|y'_j\|^\Gamma \leq \mathrm{wFV}^i_{R,\Gamma}(\|c'_j\|^\Gamma) + 1$$

and let $y_j \coloneqq h_j y'_j$. The $+1$-term is needed because we cannot assume that the infimum in the definition of wFV_R is attained. Then $\partial y_j = h_j \partial y'_j = h_j h_j^{-1} c_j = c_j$, hence $y \coloneqq y_1 + \cdots + y_m$ is a filling of c. By Remark 3.8, we may replace y by

a reduced filling without increasing either norm. Because $h_j \in \operatorname{supp}_\Gamma(c_j)$, there is some $v \in \operatorname{supp}_R(c_j)$ such that $v = h_j b_j$ for some $R\Gamma$-basis element b_j. Then $b_j \in \operatorname{supp}_R(c'_j)$, hence by Corollary 3.6

$$\|c'_j\|^\Gamma \leq (\omega_\Gamma(b_j) + 2K_{i-1}\|c_j\|)\|c_j\| = (1 + 2K_{i-1}\|c_j\|)\|c_j\| \leq (1 + 2K_{i-1}\|c\|)\|c\|.$$

It follows that

$$\|y'_j\|^\Gamma \leq \mathrm{wFV}^i_{R,\Gamma}(\|c'_j\|^\Gamma) + 1 \leq \mathrm{wFV}^i_{R,\Gamma}((1 + 2K_{i-1}\|c\|)\|c\|) + 1 = A^i_\Gamma(\|c\|).$$

Because R is 1-separated, we have $1 + \ell_\Gamma(h_j) \leq \|c_j\|^\Gamma \leq \|c\|^\Gamma$. Hence,

$$\|y_j\|^\Gamma \leq (1 + \ell_\Gamma(h_j))\|y'_j\|^\Gamma \leq \|c\|^\Gamma A^i_\Gamma(\|c\|).$$

In total, we obtain that

$$\|y\|^\Gamma \leq \sum_{j=1}^m \|y_j\|^\Gamma \leq \|c\|\|c\|^\Gamma A^i_\Gamma(\|c\|)$$

and

$$\|y\| \leq \sum_{j=1}^m \|y_j\| = \sum_{j=1}^m \|y'_j\| \leq \sum_{j=1}^m \|y'_j\|^\Gamma \leq \|c\| A^i_\Gamma(\|c\|). \qquad \square$$

We now give a criterion for the finiteness of weighted filling functions in terms of the finiteness of the non-weighted functions. The proof is based on the constructions in the proofs of [JR09, Corollary 2.5] and [BS23, Lemma 6.7].

Proposition 4.6. *Assume that R is 1-separated. For $2 \leq i \leq n$ there are constants $A_i, B_i > 0$ such that*

(1) $\mathrm{FV}^i_{R,\Gamma}(x) \leq x \cdot \mathrm{wFV}^i_{R,\Gamma}(x + A_i \cdot x^2) + x$,
(2) $\mathrm{wFV}^i_{R,\Gamma}(x) \leq B_i \cdot (\mathrm{FV}^i_{R,\Gamma}(x) + 1)^2 + x \cdot (\mathrm{FV}^i_{R,\Gamma}(x) + 1)$.

In particular, $\mathrm{FV}^i_{R,\Gamma}$ is finite if and only if $\mathrm{wFV}^i_{R,\Gamma}$ is finite.

Proof. Let $f(x) = \mathrm{FV}^i_{R,\Gamma}(x) + 1$ and let $c \in L_{i-1}$ be an $(i-1)$-cycle. Choose a filling $b = \sum_u b_u u \in L_i$ of c such that $\|b\| \leq f(\|c\|)$. By Remark 3.8, we may assume that b is reduced. By Lemma 3.9, there exist $v_u \in \operatorname{supp}_R(c)$ such that

$$\omega_\Gamma(u) \leq \omega_\Gamma(v_u) + C_i\|b\|_{\mathrm{d}} \leq \|c\|^\Gamma + C_i\|b\|,$$

where in the second inequality we use that R is 1-separated. Hence,

$$\begin{aligned}\|b\|^\Gamma = \sum_{u \in \operatorname{supp}_R(b)} |b_u|\,\omega_\Gamma(u) &\leq \|b\|(\|c\|^\Gamma + C_i\|b\|) \\ &\leq f(\|c\|)\|c\|^\Gamma + C_i f(\|c\|)^2 \\ &\leq f(\|c\|^\Gamma)\|c\|^\Gamma + C_i f(\|c\|^\Gamma)^2.\end{aligned}$$

We conclude that $\mathrm{wFV}^i_{R,\Gamma}(x) \leq C_i f(x)^2 + x \cdot f(x)$.

Let $g(x) = \mathrm{wFV}^i_{R,\Gamma}(x)$ and $c \in L_{i-1}$ be a cycle. If $g(x)$ is infinite there is nothing to show. Otherwise, consider the filling y of c from Lemma 4.5. It satisfies $\|y\| \leq \|c\| g(\|c\| + 2K_{i-1}\|c\|^2) + \|c\|$ and thus,

$$\mathrm{FV}^i_{R,\Gamma}(x) \leq x \cdot g(x + 2K_{i-1}x^2) + x. \qquad \square$$

Remark 4.7. For $n = 1$, the function $\mathrm{FV}^1_{R,\Gamma}$ is finite if and only if Γ is finite. This is because $\mathrm{FV}_R(\gamma - 1)$ grows like $\ell_\Gamma(\gamma)$ for $\gamma \in \Gamma$. In contrast, $\mathrm{wFV}^1_{R,\Gamma}$ is always at most quadratic, as $\|\gamma - 1\|^\Gamma$ also grows linearly in $\ell_\Gamma(\gamma)$.

The following weighted analogue of Theorem 4.3 is an immediate consequence of Proposition 4.6.

Theorem 4.8. *Let $n \geq 2$ and let Γ be a group of type $\mathrm{FP}_n(R)$. Assume that the norm on R is ε-separated for some $\varepsilon > 0$. Then $\mathrm{wFV}^n_{R,\Gamma}$ takes finite values.*

4.4. **A non-finite example.** In [BKV25], the authors ask the following question about the finiteness of filling functions.

Question ([BKV25, Question 2.10])**.** For which norms $|\cdot|$ does the nth homological filling function take finite values?

We conclude this section with a short example, answering this question in the negative for all normed rings which are not ε-separated. This justifies why ε-separation is a natural condition in the context of filling functions if one wants $\mathrm{FV}^n_{R,\Gamma}$ to yield a finite invariant.

Example 4.9. Assume that R is not ε-separated for any $\varepsilon > 0$. Then there exist a strictly increasing sequence $n_i \in \mathbb{N}$ and elements $r_i \in R$ such that $\frac{1}{n_i+1} < |r_i| \leq \frac{1}{n_i}$.

Let $\Gamma = \mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$ and X be the universal cover of its presentation complex, that is, $X = \mathbb{R}^2$ with $X^{(1)} = \mathbb{Z}^2$. Consider the 1-cycles $\alpha_i = r_i \cdot [x^{n_i}, y^{n_i}]$ in X. They satisfy $\|\alpha_i\| = |r_i| \cdot 4n_i \leq 4$ and $\mathrm{FV}_R(\alpha_i) = |r_i| \cdot {n_i}^2 > \frac{n_i^2}{n_i+1} > \frac{n_i}{2}$. In particular, $\mathrm{FV}^2_{R,\mathbb{Z}^2}(4) \geq \frac{n_i}{2}$ for all i, so $\mathrm{FV}^2_{R,\mathbb{Z}^2}$ is not finite.

Remark 4.10. Martínez-Pedroza and Vizcaíno Torres [MPT26] construct further examples showing that $\mathrm{FV}^n_{R,G}$ need not be finite when R is not ε-separated. In particular, for certain normed rings containing $\mathbb{Q}$, they show that $\mathrm{FV}^n_{R,G}$ is finite if and only if it is linear. Such examples exist for every $n \geq 2$.

Conversely, Theorem 4.3 shows that every ε-separated norm yields finite filling functions in every dimension $n \geq 2$ in which Γ is of type $\mathrm{FP}_n(R)$.

5. Quasi-isometry invariance of filling functions

The goal of this section is to prove quasi-isometry invariance of $\mathrm{FV}^n_{R,\Gamma}$ and $\mathrm{wFV}^n_{R,\Gamma}$ for groups of type $\mathrm{FP}_n(R)$. The proofs in this section build upon the proofs in [BKV25, Section 2]. By imposing additional restrictions on chain maps and homotopies, we can show quasi-isometry invariance without having to assume that the relevant filling functions take finite values.

Definition 5.1. Let G and H be finitely generated groups. Let L^G and L^H be finitely generated free RG- and RH-modules equipped with finite RG- and RH-bases B^G and B^H, respectively. Let $p\colon G \to H$ be a map. An R-linear map $\varphi\colon L^G \to L^H$ is *p-admissible* if there are constants $A, C \geq 0$ such that

(i) $\|\varphi(gb)\| \leq A$;
(ii) $\mathrm{supp}_H(\varphi(gb)) \subseteq B_H(p(g), C)$,

for every $g \in G$ and $b \in B^G$.

Lemma 5.2. *Let φ be p-admissible with constants A, C. Then*

(1) $\|\varphi(x)\| \leq A\|x\|$ *for every* $x \in L^G$.

Assume, in addition, that there is a constant $P \geq 0$ such that $\omega_H(p(g)) \leq P \cdot \omega_G(g)$ for every $g \in G$. Then

(2) $\|\varphi(x)\|^H \leq A(P+C)\|x\|^G$ *for every* $x \in L^G$.

Proof. For every R-basis element gb and every $h \in \operatorname{supp}_H(\varphi(gb))$, Definition 5.1 (ii) implies that $d_H(h, p(g)) \leq C$. Hence,

$$\begin{aligned}\omega_H(h) = 1 + \ell_H(h) &\leq 1 + \ell_H(p(g)) + C \\ &\leq P \cdot \omega_G(g) + C \\ &\leq (P+C) \cdot \omega_G(g),\end{aligned}$$

so $\|\varphi(gb)\|^H \leq A(P+C)\|gb\|^G$. Both statements now follow from the triangle inequality and the R-linearity of φ. □

Lemma 5.3. *Let $p, p' \colon G \to H$ be maps between finitely generated groups. Then the following hold.*

(1) *Sums and central scalar multiples of p-admissible maps are p-admissible.*
(2) *If p and p' are K-close for some $K > 0$, then a map is p-admissible if and only if it is p'-admissible.*
(3) *If Q is finitely generated, $q \colon H \to Q$ is coarsely K-Lipschitz for some $K > 0$, and φ and ψ are composable p-admissible and q-admissible maps, respectively, then $\psi \circ \varphi$ is $(q \circ p)$-admissible.*

Proof. Ad (1). Let φ and ψ be p-admissible with constants A, C and A', C', respectively. Then $\|\varphi(x) + \psi(x)\| \leq A\|x\| + A'\|x\| = (A+A')\|x\|$ and $\operatorname{supp}_H(\varphi(x)+\psi(x)) \subseteq \operatorname{supp}_H(\varphi(x)) \cup \operatorname{supp}_H(\psi(x))$, so $\operatorname{supp}_H((\varphi+\psi)(gb)) \subseteq B_H(p(g), \max(C, C'))$. For every $r \in R$, $\|r \cdot \varphi(x)\| \leq |r|\,\|\varphi(x)\| \leq |r|\,A\|x\|$ and $\operatorname{supp}_H(r \cdot \varphi(x)) \subseteq \operatorname{supp}_H(\varphi(x))$. The requirement that r lie in the centre of R ensures that $r \cdot \varphi$ is R-linear, since $(r \cdot \varphi)(ax) = ra\varphi(x) = ar\varphi(x) = a(r \cdot \varphi)(x)$ for all $a \in R$.

Ad (2). This follows immediately from the fact that $B_H(p(g), C)$ is contained in $B_H(p'(g), C + K)$.

Ad (3). Let A, C and A', C' be the constants for φ and ψ, respectively. By Lemma 5.2 (1),

$$\|\psi \circ \varphi(gb)\| \leq A'\|\varphi(gb)\| \leq A'A.$$

Every $u \in \operatorname{supp}_Q(\psi \circ \varphi(gb))$ lies in $B_Q(q(h), C')$ for some $h \in \operatorname{supp}_H(\varphi(gb))$. Since $d_H(h, p(g)) \leq C$, we have

$$\operatorname{supp}_Q(\psi \circ \varphi(gb)) \subseteq B_Q(q(p(g)), C' + KC + K).$$

Thus, $\psi \circ \varphi$ is $(q \circ p)$-admissible. □

Remark 5.4. Let Γ be a finitely generated group. Every equivariant R-linear map between finitely generated free $R\Gamma$-modules equipped with finite bases is id_Γ-admissible.

Lemma 5.5. *Let $k \geq 0$ and let Γ be a finitely generated group. Let $L_* \to R$ be a free $R\Gamma$-resolution such that L_i is finitely generated for $0 \leq i \leq k+1$, and fix finite $R\Gamma$-bases B_i in these degrees. For every $r \geq 0$ there are constants $D_r, \rho_r \geq 0$ such that every k-cycle $c \in L_k$ satisfying $\operatorname{supp}_\Gamma(c) \subseteq B_\Gamma(g, r)$ for some $g \in \Gamma$ admits an element $y \in L_{k+1}$ such that*

(1) $\partial y = c$;
(2) $\|y\| \leq D_r\|c\|$;
(3) $\operatorname{supp}_\Gamma(y) \subseteq B_\Gamma(g, \rho_r)$.

Proof. The augmented chain complex $L_* \twoheadrightarrow R$ admits an R-linear chain contraction t_*. It satisfies $\partial t_k(c) = c$ for every k-cycle c. The R-support $\operatorname{supp}_R(g^{-1}c)$ is contained in the finite set $B_\Gamma(1_\Gamma, r)B_k$. Hence, there are constants $D_r, \rho_r \geq 0$ such that $\|t_k(v)\| \leq D_r$ and $\operatorname{supp}_\Gamma(t_k(v)) \subseteq B_\Gamma(1_\Gamma, \rho_r)$ for every $v \in B_\Gamma(1_\Gamma, r)B_k$. Since $g^{-1}c$ is a cycle, the chain $y \coloneqq gt_k(g^{-1}c)$ fills c. Moreover, $\|y\| \leq D_r\|c\|$ and $\operatorname{supp}_\Gamma(y) \subseteq B_\Gamma(g, \rho_r)$, as required. □

For the remainder of this section, let R be a normed ring, let $n \geq 1$, and let G and H be groups of type $\mathrm{FP}_n(R)$. Fix free resolutions $L_*^G \to R$ and $L_*^H \to R$ as in Remark 2.8, together with finite bases B_i^G and B_i^H in degrees $0 \leq i \leq n$.

Lemma 5.6. *Let $0 \leq k \leq n-1$ and let $p\colon G \to H$ be a map. Let F be a finitely generated free RG-module equipped with a finite RG-basis B. Suppose that $\alpha\colon F \to L_k^H$ is p-admissible and $\operatorname{im}(\alpha) \subseteq \ker(\partial_k^H)$. Then there is an R-linear p-admissible map $\beta\colon F \to L_{k+1}^H$ satisfying $\partial_{k+1}\beta = \alpha$.*

Proof. Let A, C be the constants showing that α is p-admissible. Let D, ρ be the constants supplied by Lemma 5.5 for $r = C$. For each $gb \in GB$, choose a filling y_{gb} of $\alpha(gb)$ satisfying $\|y_{gb}\| \leq D\|\alpha(gb)\| \leq DA$ and $\operatorname{supp}_H(y_{gb}) \subseteq B_H(p(g), \rho)$. Defining $\beta(gb) \coloneqq y_{gb}$ and extending R-linearly gives a p-admissible map with $\partial_{k+1}\beta = \alpha$. □

Lemma 5.7. *Let $f\colon G \to H$ be a (K, K)-quasi-isometry. Then there are R-linear maps $f_i\colon L_i^G \to L_i^H$ and constants $D_i, \widetilde{D}_i \geq 1$ for $0 \leq i \leq n$ such that*

(1) *f_0 is the R-linear extension of f;*
(2) *$f_{i-1}\partial = \partial f_i$ for $1 \leq i \leq n$;*
(3) *f_i is f-admissible for $0 \leq i \leq n$;*
(4) *$\|f_i(x)\| \leq D_i\|x\|$ and $\|f_i(x)\|^H \leq \widetilde{D}_i\|x\|^G$ for every $x \in L_i^G$ and $0 \leq i \leq n$.*

Proof. Let $f_0\colon RG \to RH$ be the R-linear extension of f. Then f_0 is f-admissible and satisfies $\partial_0 f_0 = f_{-1}\partial_{-1}$, where $\partial_{-1} = \varepsilon$ and $f_{-1} = \mathrm{id}_R$.

For $1 \leq i \leq n$, suppose inductively that maps f_j have been constructed for $j < i$. By Lemma 5.3 and Remark 5.4, the map $f_{i-1}\partial_i\colon L_i^G \to L_{i-1}^H$ is f-admissible. It takes values in cycles because $\partial_{i-1}f_{i-1}\partial_i = f_{i-2}\partial^2 = 0$. Hence, Lemma 5.6 gives an f-admissible map $f_i\colon L_i^G \to L_i^H$ satisfying $\partial f_i = f_{i-1}\partial$. Thus, induction on i produces the required partial chain map.

The bounds on the norms follow from Lemma 5.2 together with the inequality

$$1 + \ell_H(f(g)) \leq 1 + K\ell_G(g) + K + \ell_H(f(1_G)) \leq P(1 + \ell_G(g)),$$

where $P \coloneqq K + \ell_H(f(1_G)) + 1$. □

Remark 5.8. The map f_1 in Lemma 5.7 can be described explicitly. Let $S \cong B_1^G$ and $T \cong B_1^H$ be finite generating sets for G and H, respectively. Let $gb \in GB_1^G$ be an R-basis element corresponding to an edge from g to gs in the Cayley graph $\mathrm{Cay}(G, S)$. We may then choose $f_1(gb)$ to be a chain representing a path from $f(g)$ to $f(gs)$ in the Cayley graph $\mathrm{Cay}(H, T)$.

Lemma 5.9. *Let $f\colon G \to H$ and $h\colon H \to G$ be (K, K)-quasi-isometries such that $h \circ f$ is K-close to id_G. Let f_* and h_* be the partial chain maps obtained from*

Lemma 5.7. Then there are R-linear maps $s_i\colon L_i^G \to L_{i+1}^G$ and constants $E_i, \widetilde{E}_i \geq 1$ for $0 \leq i \leq n-1$ such that

(1) $\partial s_i + s_{i-1}\partial = \mathrm{id} - h_i \circ f_i$ *for* $0 \leq i \leq n-1$;
(2) s_i *is* id_G*-admissible for* $0 \leq i \leq n-1$;
(3) $\|s_i(x)\| \leq E_i\|x\|$ *and* $\|s_i(x)\|^G \leq \widetilde{E}_i\|x\|^G$ *for every* $x \in L_i^G$ *and* $0 \leq i \leq n-1$.

Proof. By Lemma 5.3 (2) and (3), each $h_i \circ f_i$ is $(h \circ f)$-admissible and hence id_G-admissible. Therefore, Lemma 5.3 (1) shows that the maps $\psi_i := \mathrm{id} - h_i \circ f_i$ are all id_G-admissible.

Set $s_{-1} = 0$. Since $\varepsilon\psi_0 = \varepsilon(\mathrm{id} - h_0 \circ f_0) = \mathrm{id}_R - \mathrm{id}_R = 0$, the map ψ_0 takes values in cycles. Thus, Lemma 5.6 gives an id_G-admissible map $s_0\colon L_0^G \to L_1^G$ satisfying $\partial s_0 = \psi_0$.

For $1 \leq i \leq n-1$, suppose inductively that id_G-admissible maps s_j have been constructed for $j < i$ and satisfy $\partial s_j + s_{j-1}\partial = \psi_j$. By Lemma 5.3 and Remark 5.4, the map $\Psi_i := \psi_i - s_{i-1}\partial_i\colon L_i^G \to L_i^G$ is id_G-admissible. Moreover,

$$\begin{aligned}\partial\Psi_i &= \psi_{i-1}\partial_i - \partial s_{i-1}\partial_i \\ &= \psi_{i-1}\partial_i - (\psi_{i-1} - s_{i-2}\partial)\partial_i = 0.\end{aligned}$$

Hence, Lemma 5.6 gives an id_G-admissible map $s_i\colon L_i^G \to L_{i+1}^G$ satisfying $\partial s_i = \Psi_i$. As in Lemma 5.7, the bounds on the norms follow from Lemma 5.2. □

Theorem 5.10. *Suppose that $n \geq 2$ and that G and H are quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{FV}_{R,G}^n \approx \mathrm{FV}_{R,H}^n$.*

Proof. Let $f\colon G \to H$ be a quasi-isometry with quasi-inverse $h\colon H \to G$. Let f_* and h_* be the partial chain maps obtained by applying Lemma 5.7 to f and h, respectively. Denote the associated constants by D_i and D_i'. Let s_i and E_i be the homotopy and constant supplied by Lemma 5.9.

Let $z \in L_{n-1}^G$ be an $(n-1)$-cycle. Then $f_{n-1}(z)$ is a cycle and $\|f_{n-1}(z)\| \leq D_{n-1}\|z\|$. Let $b \in L_n^H$ be any filling of $f_{n-1}(z)$. Then $h_n(b) + s_{n-1}(z)$ is a filling of z, as

$$\begin{aligned}\partial h_n(b) + \partial s_{n-1}(z) &= h_{n-1}(\partial b) + \partial s_{n-1}(z) \\ &= h_{n-1}(f_{n-1}(z)) + (z - h_{n-1}(f_{n-1}(z)) - s_{n-2}(\partial z)) = z.\end{aligned}$$

Moreover,

$$\|h_n(b) + s_{n-1}(z)\| \leq D_n'\|b\| + E_{n-1}\|z\|.$$

Taking the infimum over all fillings b gives

$$\mathrm{FV}_R(z) \leq D_n'\,\mathrm{FV}_R(f_{n-1}(z)) + E_{n-1}\|z\|,$$

and taking the supremum over all cycles with norm at most l yields

$$\mathrm{FV}_{R,G}^n(l) \leq D_n'\,\mathrm{FV}_{R,H}^n(D_{n-1}l) + E_{n-1}l. \tag{5.1}$$

Note that this inequality remains valid if a filling function takes the value ∞. This proves $\mathrm{FV}_{R,G}^n \preccurlyeq \mathrm{FV}_{R,H}^n$. The argument is symmetric, completing the proof. □

Remark 5.11. It was proven by Alonso [Alo94] that $\mathrm{FP}_n(R)$ is a quasi-isometry invariant. We could therefore weaken the hypothesis of Theorems 5.10 and 5.13 to just requiring G to be of type $\mathrm{FP}_n(R)$.

Remark 5.12. The same argument as in Theorem 5.10 can be used to obtain quasi-isometry invariance of filling functions $\mathrm{FV}^n_{R,X}$ of metric simplicial complexes under suitable assumptions on the complexes. For a metric simplicial complex X, let $\Sigma_i(X)$ denote its set of i-simplices, with a chosen orientation for each simplex. Thus, $C_i(X;R) \cong R[\Sigma_i(X)]$ as a free R-module. Equip $C_i(X;R)$ with a coordinate function $\lambda_i^X \colon \Sigma_i(X) \to X$ with $\lambda_i^X(\sigma) \in \sigma$, choosing, for example, the barycentre of each simplex. For $c \in C_i(X;R)$, set $\mathrm{supp}_X(c) \coloneqq \lambda_i^X(\mathrm{supp}_R(c))$. Let $p \colon X \to Y$ be a map. Call an R-linear map $\varphi \colon C_i(X;R) \to C_j(Y;R)$ *p-admissible* if there are constants $A, C \geq 0$ such that $\|\varphi(\sigma)\| \leq A$ and $\mathrm{supp}_Y(\varphi(\sigma)) \subseteq B_Y(p(\lambda_i^X(\sigma)), C)$ for every $\sigma \in \Sigma_i(X)$.

Suppose that simplices have uniformly bounded diameters and that the analogue of Lemma 5.5 holds in degrees $0 \leq k < n$, that is, every k-cycle c supported in $B_X(x,r)$ admits a filling b with $\|b\| \leq D_r\|c\|$ and $\mathrm{supp}_X(b) \subseteq B_X(x, \rho_r)$, for some constants D_r, ρ_r. Under these assumptions on both X and Y, the same constructions from this section yield admissible chain maps and homotopies from a quasi-isometry $X \to Y$ and its quasi-inverse. Consequently, for $n \geq 2$, the corresponding nth filling functions are $\approx$-equivalent.

For example, the above assumptions are satisfied by every $(n-1)$-R-acyclic Rips complex $X = \mathrm{Rips}_s(G)$ of a finitely generated group G, equipped with its standard piecewise Euclidean metric.

The following theorem is the weighted version of Theorem 5.10.

Theorem 5.13. *Suppose that $n \geq 2$ and that G and H are quasi-isometric groups of type $\mathrm{FP}_n(R)$. Then $\mathrm{wFV}^n_{R,G} \approx \mathrm{wFV}^n_{R,H}$.*

Proof. Repeat the proof of Theorem 5.10 using the weighted bounds $\widetilde{D}_i$, $\widetilde{D}'_i$, and $\widetilde{E}_i$ supplied by Lemmas 5.7 and 5.9. □

Even though there is a close relationship between weighted and unweighted filling functions due to Proposition 4.6, it is unclear whether knowledge of one of the functions allows us to recover the other.

Question 5.14. Are there groups G, H of type $\mathrm{FP}_n(R)$ such that $\mathrm{FV}^n_{R,G} \approx \mathrm{FV}^n_{R,H}$ but $\mathrm{wFV}^n_{R,G} \not\approx \mathrm{wFV}^n_{R,H}$ or vice versa?

6. Cohomology

This section provides an algebraic proof of the quasi-isometry invariance of group cohomology with coefficients in the group ring, originally established by Gersten [Ger93] for finitely presented groups and, for commutative coefficient rings, more generally by Li [Li17] for groups of type $\mathrm{FP}(R)$. The argument presented here follows Gersten's proof in spirit without relying on the groups being finitely presented. As a consequence, we deduce quasi-isometry invariance of the cohomological dimension $\mathrm{cd}_R(G)$ for groups of type $\mathrm{FP}(R)$, as well as of being a (Poincaré) duality group when R is commutative. For commutative coefficient rings, these results are due to Sauer [Sau06] and Li [Li17], respectively. The proof uses the admissible chain maps and homotopies constructed in Section 5.

The *cohomological dimension* $\mathrm{cd}_R(G)$ of a group G is defined as the smallest length n such that there is a projective RG-resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R$$

of the trivial RG-module R. It is ∞ if no such resolution of finite length exists. If G is of type $\mathrm{FP}(R)$, the above definition is equivalent to

$$\mathrm{cd}_R(G) = \max\{i \geq 0 \mid H^i(G; RG) \neq 0\}.$$

See [Bro12, Proposition VIII.6.7] for a proof.

For our purposes, it is convenient to use the following characterization of (Poincaré) duality groups (see [Bie81, Chapter III, Theorem 9.2]). If R is commutative, a group G is called a *duality group of dimension n over R* if

(i) G is of type $\mathrm{FP}(R)$;
(ii) $H^k(G; RG) = 0$ for $k \neq n$;
(iii) $H^n(G; RG)$ is flat as an R-module.

The module $D := H^n(G; RG)$ is called the *dualising module*. Such a group is called a *Poincaré duality group of dimension n over R* if $D \cong R$ as R-modules.

In the following let G and H be groups of type $\mathrm{FP}(R)$ for a ring R. Furthermore, let L^G_* and L^H_* be free RG- and RH-resolutions of R, respectively, such that all L^G_i and L^H_i are finitely generated. Let B^G_i and B^H_i be choices of RG- and RH-bases of L^G_i and L^H_i, respectively.

Definition 6.1. Let $\varphi\colon L \to L'$ be an R-linear map between free R-modules with bases B and B' respectively. We say that φ is *algebraically proper* if for each $y \in B'$ the set

$$I_\varphi(y) := \{x \in B \mid y \in \mathrm{supp}_R(\varphi(x))\}$$

is finite.

Lemma 6.2. *Let $p\colon G \to H$ be a map such that preimages of finite subsets are finite. Every p-admissible map $\varphi\colon RG^n \to RH^m$ is algebraically proper with respect to the standard R-bases.*

Proof. Let B_n and B_m denote the standard RG- and RH-bases of RG^n and RH^m, respectively. Thus, GB_n and HB_m are the corresponding R-bases. Let C be a constant such that $\mathrm{supp}_H(\varphi(gb)) \subseteq B_H(p(g), C)$ for every $gb \in GB_n$.

Let $y = hb' \in HB_m$ and suppose that $x = gb \in I_\varphi(y)$. By definition of $I_\varphi(y)$, this means that hb' lies in $\mathrm{supp}_R(\varphi(gb))$. In particular, $h \in \mathrm{supp}_H(\varphi(gb))$. Since φ is p-admissible, we have

$$h \in \mathrm{supp}_H(\varphi(gb)) \subseteq B_H(p(g), C).$$

Thus, $d_H(p(g), h) \leq C$, so $p(g) \in B_H(h, C)$ and therefore $g \in p^{-1}(B_H(h, C))$. It follows that $x = gb \in p^{-1}(B_H(h, C))B_n$, so

$$I_\varphi(y) \subseteq p^{-1}(B_H(h, C))B_n,$$

which is finite by the assumption on p. □

Corollary 6.3. *Let $f\colon G \to H$ be a quasi-isometry. There exists an R-linear chain homotopy equivalence $f_*\colon L^G_* \to L^H_*$ with chain homotopy inverse $h_*\colon L^H_* \to L^G_*$ and homotopies $s_*\colon h_* \circ f_* \simeq \mathrm{id}$, $t_*\colon f_* \circ h_* \simeq \mathrm{id}$ such that f_i, h_i, s_i and t_i are algebraically proper for all i.*

Proof. Equip R with the discrete norm and let h be a quasi-inverse of f. Since G and H are of type $\mathrm{FP}(R)$, the constructions from Lemmas 5.7 and 5.9 can be carried out in every degree. They give f-admissible and h-admissible chain maps f_* and h_*, respectively, and id_G-admissible and id_H-admissible homotopies s_* and

t_*, respectively. Quasi-isometries and identity maps have finite preimages of finite subsets, so all these maps are algebraically proper by Lemma 6.2. □

For a group Γ and a left $R\Gamma$-module M consider the right $R\Gamma$-module $\hom^{\mathrm{c}}_R(M,R)$ consisting of those R-linear maps $f\colon M\to R$ such that, for every $m\in M$, we have $f(\gamma m)=0$ for all but finitely many $\gamma\in\Gamma$. By [Bro12, Lemma VIII.7.4], there is a natural isomorphism

$$\hom_{R\Gamma}(M,R\Gamma)\cong\hom^{\mathrm{c}}_R(M,R). \tag{6.1}$$

If M is a free $R\Gamma$-module with finite $R\Gamma$-basis B, the module $\hom^{\mathrm{c}}_R(M,R)$ consists of those maps f for which $f(\gamma b)\neq 0$ for only finitely many $\gamma b\in\Gamma B$.

Evidently, an algebraically proper map $\varphi\colon RG^n\to RH^m$ induces a well-defined map $\hom^{\mathrm{c}}_R(RH^m,R)\to\hom^{\mathrm{c}}_R(RG^n,R)$ through precomposition.

Theorem 6.4 ([Li17, Corollary 4.41])**.** *Let G and H be groups of type $\mathrm{FP}(R)$. If G and H are quasi-isometric then $H^i(G;RG)\cong H^i(H;RH)$ as right R-modules for all $i\geq 0$ and any ring R.*

Proof. The chain homotopy equivalence from Corollary 6.3 induces a well-defined R-linear chain homotopy equivalence $\hom^{\mathrm{c}}_R(L^H_*,R)\to\hom^{\mathrm{c}}_R(L^G_*,R)$. Thus,

$$\begin{aligned}H^i(G;RG)&\cong H^i(\hom_{RG}(L^G_*,RG))\cong H^i(\hom^{\mathrm{c}}_R(L^G_*,R))\\&\cong H^i(\hom^{\mathrm{c}}_R(L^H_*,R))\cong H^i(\hom_{RH}(L^H_*,RH))\cong H^i(H;RH),\end{aligned}$$

where the second and fourth isomorphisms use Equation (6.1). □

Corollary 6.5 ([Sau06, Theorem 1.2])**.** *The cohomological dimension $\mathrm{cd}_R(G)$ is a quasi-isometry invariant of groups of type $\mathrm{FP}(R)$.*

Another consequence of Theorem 6.4 is the quasi-isometry invariance of being a (Poincaré) duality group. The proof is the same as that of [Ger93, Corollary 3].

Corollary 6.6 ([Ger93, Corollary 3; Li17, Corollary 4.48])**.** *Let R be a commutative ring and let G and H be quasi-isometric groups of type $\mathrm{FP}(R)$. If G is a duality group, respectively a Poincaré duality group over R, then so is H.*

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