

From finiteness properties to polynomial filling via homological algebra

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Finiteness properties

Definition

A $\mathbb{Z}[G]$ -Module M is of type FP_n if there is a partial $\mathbb{Z}[G]$ -resolution

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

consisting of finitely generated projective modules.

The group G is of type FP_n if \mathbb{Z} is FP_n as a $\mathbb{Z}[G]$ -module.

Remark

- We may replace *projective* by *free*.
- If G is finitely presented and of type FP_n there exists a contractible free G -CW-complex with cocompact n -skeleton.

2-out-of-3 for FP_n (Bieri)

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of $\mathbb{Z}[G]$ -modules.

- If A is FP_{n-1} and B is FP_n , then C is FP_n .
- If A is FP_n and C is FP_n , then B is FP_n .
- If B is FP_n and C is FP_{n+1} , then A is FP_n .

What we want to generalize

Theorem (Brown)

Assume X is an $(n - 1)$ -acyclic G -CW-complex with cocompact n -skeleton such that each stabilizer G_σ of a d -cell σ is of type FP_{n-d} . Then G is of type FP_n .

Proof

$C_d := C_d(X) \cong \bigoplus_{\sigma \in I_d} \mathbb{Z}[G/G_\sigma]$ is of type FP_{n-d} .

$$\begin{array}{ccccccc} \text{FP}_0 & & \text{FP}_1 & & \text{FP}_2 & & \text{FP}_n & & \text{FP}_n \\ C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & \dots & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0=\varepsilon} & \mathbb{Z} \\ & \searrow & \nearrow & \searrow & \nearrow & & \searrow & \nearrow & & & \\ & & \ker \partial_{n-1} & & \ker \partial_{n-2} & & & & \ker \partial_0 & & \\ & & \text{FP}_0 & & \text{FP}_1 & & & & \text{FP}_{n-1} & & \end{array}$$

Homological filling functions

Family of norms on $\mathbb{Q}[G/H]$

Let ℓ_G be a word length on G . We define the norms

$$\left\| \sum_{g \in G} a_g g \right\|_n^G := \sum_{g \in G} |a_g| \cdot (1 + \ell_{G/H}(g))^n.$$

This is inspired by Connes-Moscovici, Meyer, Ogle, Ramsey.

Family of norms on $\mathbb{Q}[G/H]$: $\|\sum_{g \in G} a_g g\|_n^G := \sum_{g \in G} |a_g| \cdot (1 + \ell_{G/H}(g))^n$

Definition

Let C_* be an $(n - 1)$ -acyclic $\mathbb{Z}[G]$ -chain complex consisting of permutation modules. The n -th (weighted) homological filling function of C_* is

$$\text{FV}^n(c) = \min_{\substack{b \in C_n \\ \partial b = c}} \|b\|_0^G$$

$$\text{FV}_{C_*}^n(v) = \sup_{\substack{c \in \ker \partial_{n-1} \\ \|c\|_0^G \leq v}} \text{FV}^n(c)$$

$$\text{wFV}^n(c) = \min_{\substack{b \in C_n \\ \partial b = c}} \|b\|_1^G$$

$$\text{wFV}_{C_*}^n(v) = \sup_{\substack{c \in \ker \partial_{n-1} \\ \|c\|_1^G \leq v}} \text{wFV}^n(c).$$

- For an $(n - 1)$ -acyclic G -CW-complex: $\text{FV}_X^n := \text{FV}_{C_*(X)}^n$ and $\text{wFV}_X^n := \text{wFV}_{C_*(X)}^n$.
- For a group G of type FP_n : $\text{FV}_X^n := \text{FV}_{L_*}^n$ and $\text{wFV}_X^n := \text{wFV}_{L_*}^n$ for a suitable free $\mathbb{Z}[G]$ -resolution L_* .

A version of Brown's theorem for filling functions

Definition

We say a filling function is *polynomial* if it is bounded by a polynomial.

Remark

For $n \geq 2$, wFV_G^n is polynomial if and only if FV_G^n is polynomial.

Theorem (Sauer-W)

Assume X is an $(n - 1)$ -acyclic G -CW-complex with cocompact n -skeleton such that

- wFV_X^k is polynomial for $k \leq n$

and each stabilizer G_σ of a d -cell

- is of type FP_{n-d} and $\text{wFV}_{G_\sigma}^k$ is polynomial for $k \leq n - d$;
- is at most polynomially distorted in G .

Then G is of type FP_n and wFV_G^k is polynomial for $k \leq n$.

An exact structure on $\mathbb{Z}[G]$ -modules

Definition

A *bornology* on a set X is a collection \mathcal{B} of subsets, called *bounded subsets*, such that

- \mathcal{B} covers X ,
- \mathcal{B} is stable under taking subsets and finite unions.

A map between bornological sets is *bounded* if it maps bounded sets to bounded sets.

Define *bornological group/ring/module (over a bornological ring)* as a group/ring/module such that all operations are bounded.

The Cayley bornology

Family of norms on $\mathbb{Q}[G]$: $\|\sum_{g \in G} a_g g\|_n^G := \sum_{g \in G} |a_g| \cdot (1 + \ell_G(g))^n$

Bounded subsets of $\mathbb{Q}[G]$

A subset of $\mathbb{Q}[G]$ is bounded if it is bounded with respect to $\|\cdot\|_n^G$ for every $n \in \mathbb{N}_0$.
This turns $\mathbb{Q}[G]$ into a bornological ring.

Cayley bornology on arbitrary $\mathbb{Q}[G]$ -modules

The *Cayley bornology* of a $\mathbb{Q}[G]$ -module M is the smallest bornology such that

- M is a bornological $\mathbb{Q}[G]$ -module;
- every $\mathbb{Q}[G]$ -linear map $\mathbb{Q}[G] \rightarrow M$ is bounded.

This makes every $\mathbb{Q}[G]$ -linear map between $\mathbb{Q}[G]$ -modules bounded.

An exact structure on $\mathbb{Z}[G]$ -modules

Definition

An exact sequence of $\mathbb{Z}[G]$ -modules

$$0 \longrightarrow A \overset{r}{\longleftarrow} B \overset{s}{\longrightarrow} C \longrightarrow 0$$

is E_G -exact if

- it splits via \mathbb{Z} -linear maps r , s , and
- $\mathbb{Q} \otimes_{\mathbb{Z}} r$ and $\mathbb{Q} \otimes_{\mathbb{Z}} s$ are bounded with respect to the Cayley bornology.

Theorem (Sauer-W)

With respect to the above class E_G of short exact sequence, $\mathbb{Z}[G]$ -modules form an exact category.

Finiteness properties relative to E_G

Definition

A $\mathbb{Z}[G]$ -Module M is of type $FP_n(E_G)$ if there is a partial $\mathbb{Z}[G]$ -resolution

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

consisting of finitely generated projective modules such that

$$0 \rightarrow \ker \partial_i \hookrightarrow P_i \twoheadrightarrow \operatorname{im} \partial_i \rightarrow 0$$

is E_G -exact for $0 \leq i \leq n$.

Remark

We may replace *projective* by *free*.

Theorem (Sauer-W)

- wFV_X^k is polynomial for $k \leq n \iff C_*(X) \rightarrow \mathbb{Z}$ is \mathbb{E}_G -exact up to degree $n - 1$.
- wFV_G^k is polynomial for $k \leq n \iff \mathbb{Z}$ is of type $\text{FP}_n(\mathbb{E}_G)$.

2-out-of-3 for $\text{FP}_n(\mathbb{E}_G)$ (Sauer-W)

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an \mathbb{E}_G -exact sequence of $\mathbb{Z}[G]$ -modules.

- If A is $\text{FP}_{n-1}(\mathbb{E}_G)$ and B is $\text{FP}_n(\mathbb{E}_G)$, then C is $\text{FP}_n(\mathbb{E}_G)$.
- If A is $\text{FP}_n(\mathbb{E}_G)$ and C is $\text{FP}_n(\mathbb{E}_G)$, then B is $\text{FP}_n(\mathbb{E}_G)$.
- If B is $\text{FP}_n(\mathbb{E}_G)$ and C is $\text{FP}_{n+1}(\mathbb{E}_G)$, then A is $\text{FP}_n(\mathbb{E}_G)$.

The algebraic formulation

wFV_X^k is polynomial for $k \leq n \iff C_*(X) \rightarrow \mathbb{Z}$ is E_G -exact up to degree $n - 1$.

Theorem (Sauer-W)

Assume X is a G -CW-complex with cocompact n -skeleton such that

- $C_*(X) \rightarrow \mathbb{Z}$ is E_G -exact up to degree $n - 1$,

and for each stabilizer G_σ of a d -cell

- \mathbb{Z} is of type $\text{FP}_{n-d}(E_{G_\sigma})$ and G_σ is at most polynomially distorted in G .

Then \mathbb{Z} is of type $\text{FP}_n(E_G)$.

Question

Does polynomiality of FV_X^k for $k \geq 2$ imply that $C_*(X) \rightarrow \mathbb{Z}$ is E_G -exact?

- Yes, if X is locally finite.
- Yes, if X is the Bass-Serre tree of a graph of groups. (FV_X^k is automatically linear.)

Applications

- For X being the Bass-Serre tree.
- For X being the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$, one obtains polynomial filling functions for $\mathrm{SL}_n(\mathbb{Z}[1/p])$ up to degree $n - 2$.
(cf. Leuzinger-Young, Bestvina-Eskin-Wortman)
- 2-out-of-3 properties for polynomiality of filling functions for group extensions $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$, where N is at most polynomially distorted in G
 - e.g. FV_G^n is polynomial for nilpotent groups.

Remark

The results also hold for other sufficiently nice normed coefficient rings e.g. \mathbb{F}_q equipped with the discrete norm.